which generator is parallel to the axis Oz and from below by the domain $D \in xoy$.

In the Cartesian coordinate system an element of area is as follows $d\sigma = dxdy$.

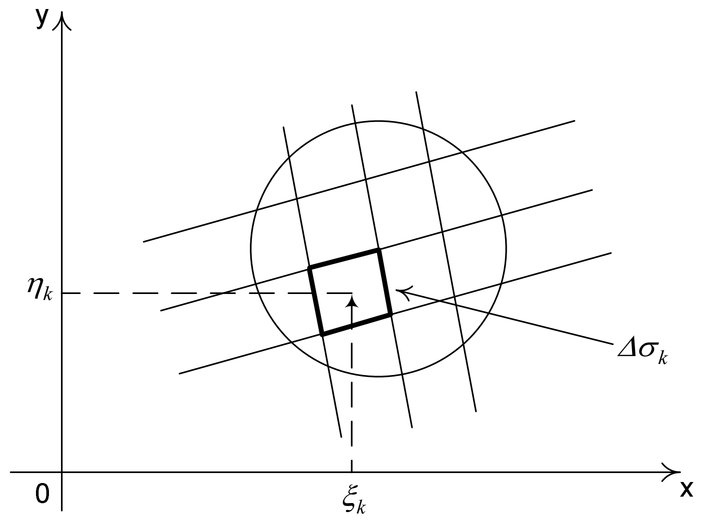


Fig. 27

$\Delta\sigma_k$ − is an elementary area; point $\xi_k;\eta_k \in \Delta\sigma_k$.

The diameter of the elementary area is called the greatest distance between two boundary points of this domain.

The properties of a double integral

1. $\iint\limits_D \left[f_1(x,y) \pm f_2(x,y) \right] d\sigma = \iint\limits_D f_1(x,y)\, d\sigma \pm \iint\limits_D f_2(x,y)\, d\sigma$

The double integral calculated from the sum or the difference of two functions is equal to the sum or the difference of the double integrals over domain D.

2. $\iint\limits_D \lambda f(x,y)\, d\sigma = \lambda \iint\limits_D f(x,y)\, d\sigma$, where $\lambda = const$.

The constant multiplier can be taken out of the sign of a double integral.

3. If $D = D_1 \cup D_2$, then $\iint\limits_D f(x,y)\, d\sigma = \iint\limits_{D_1} f(x,y)\, d\sigma + \iint\limits_{D_2} f_2(x,y)\, d\sigma$, this follows from the geometric meaning of the double integral.

Rules of calculation of double integrals

Let’s consider two kinds of integration domains D_1 (fig.18) and D_2 (fig.19).

Functions $\varphi_1(x)$ and $\varphi_2(x)$ are continuous functions having one point of intersection with straight lines $x = a$ and $x = b$.

Then $\iint_{D_1} f(x,y)dxdy = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x_1y)dy = \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x_1y)dy \right] dx$. This is formula I.

Where an integration is respectively y, but x is considered to be constant.

$D_1: \; x=a; \; x=b; \; (a<b) y=\varphi_1(x); \; y=\varphi_2(x); \; \varphi_1(x) \le \varphi_2(x)$.

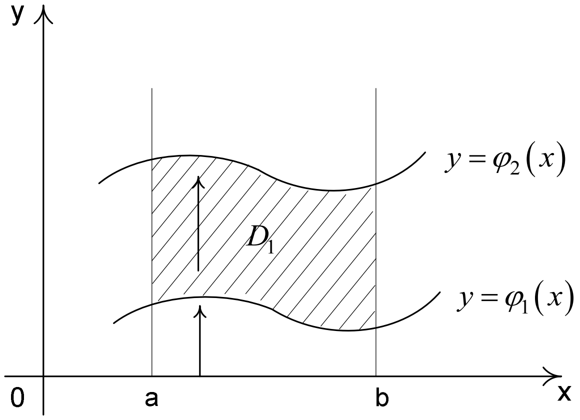


Fig.28

$D_2: \; y=c; \; y=d; \; (c<d); \; x=\psi_1(y); \; x=\psi_2(y); \; \psi_1(y) \le \psi_2(y)$.

Functions $\psi_1(y)$ and $\psi_2(y)$ are continuous ones with one point of intersection with straight lines $y=c$ and $y=d$.

Thus $\iint_{D_2} f(x,y)dxdy = \int_c^d dy \int_{\psi_1(y)}^{\psi_2(y)} f(x_1y)dx = \int_c^d \left[\int_{\psi_1(y)}^{\psi_2(y)} f(x_1y)dx \right] dy$.

This is formula II.

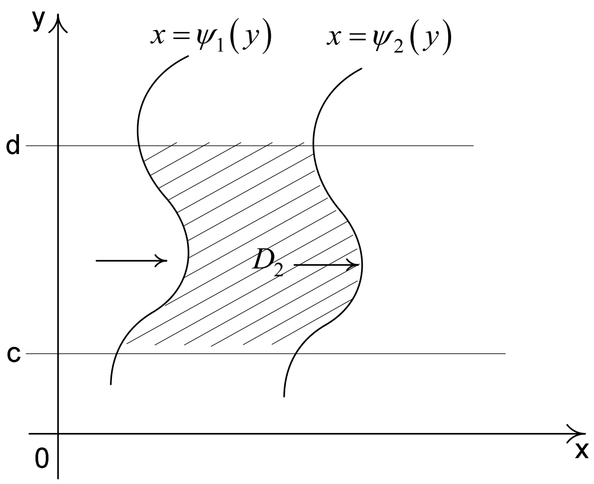
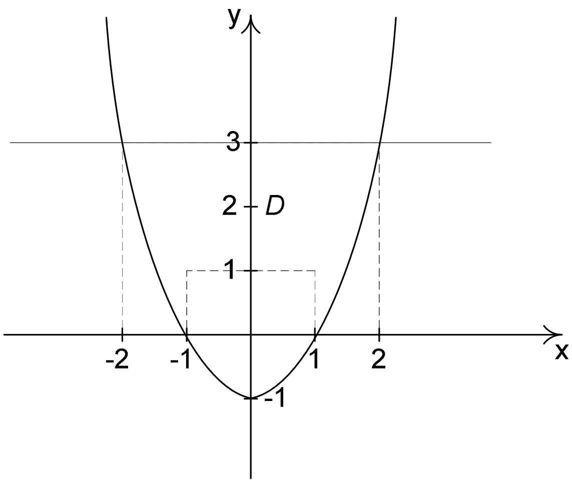


Fig.29

Where integration is respectively x, but y is considered to be constant.
The right parts of mentioned above formulae are interated or repeated integrals.
Any domain of integration can be presented as a sum of domains D_1 and D_2.
Calculating a double integral according to formula I, x changes from a to b, y changes from the below function $\varphi_1(x)$ to the above function $\varphi_2(x)$.
Calculating a double integral according to formula II, y changes c to d, x changes from the left function $\psi_1(x)$ to the right function $\psi_2(x)$.
Let's consider the examples.

Example. Let's define the limits of integration according to formula I and II if domain $D: \ y = x^2 - 1, y = 3$.

Solution. Let's make a drawing Fig.30 and find points of intersection



$3 = x^2 - 1$; $x^2 = 4$;
$x_{1,2} = \pm 2$;
$y = x^2 - 1$; $x^2 = y + 1$;
$x = \pm\sqrt{y+1}$;

Fig.30

$$\iint\limits_D f(x,y)\,dxdy = \int\limits_{-2}^{2} dx \int\limits_{x^2-1}^{3} f(x,y)\,dy = \int\limits_{-1}^{3} dy \int\limits_{-\sqrt{y+1}}^{+\sqrt{y+1}} f(x,y)\,dx .$$

Example. Let's calculate mentioned above integral according to the formula I and II if $f(x,y) = x + y$.

Solution. Outcoming from the geometric meaning of a double integral we must obtain an equal result according to formula I and II.
Let's prove this fact

$$\iint\limits_D (x,y)\,dxdy = \int\limits_{-2}^{2} dx \int\limits_{x^2-1}^{3} f(x,y)\,dy = \int\limits_{-2}^{2} \left[xy \Big|_{x^2-1}^{3} + \frac{y^2}{2} \Big|_{x^2-1}^{3} \right] dx =$$

$$
=\int_{-2}^{2}\left(3x-x^2+x+\frac{9}{2}-\frac{x^4-2x^2+1}{2}\right)dx=\int_{-2}^{2}\left(4x-x^3+4-\frac{x^4}{2}+x^2\right)dx=
$$

$$
=\underset{0}{\underset{\parallel}{\cancel{4\frac{x^2}{2}\Big|_{-2}^{2}}}}-\underset{0}{\underset{\parallel}{\cancel{\frac{x^4}{4}\Big|_{-2}^{2}}}}+4x\Big|_{-2}^{2}-\frac{x^5}{10}\Big|_{-2}^{2}+\frac{x^3}{3}\Big|_{-2}^{2}=16-\frac{64}{10}+\frac{16}{3}=\frac{224}{15}.
$$

According to Formula II:

$$
\iint_D (x+y)\,dxdy=\int_{-1}^{3}dy\int_{-\sqrt{y+1}}^{\sqrt{y+1}}(x+y)\,dx=\int_{-1}^{3}\left[\underset{0}{\underset{\parallel}{\frac{x^2}{2}\Big|_{-\sqrt{y+1}}^{\sqrt{y+1}}}}+yx\Big|_{-\sqrt{y+1}}^{\sqrt{y+1}}\right]dy=
$$

$$
=2\int_{-1}^{3}y\sqrt{y+1}\,dy=\left|\begin{array}{l}\sqrt{y+1}=t\\ y=t^2-1\\ dy=2tdt\\ y=-1;\ t=0\\ y=3;\quad t=2\end{array}\right|=2\cdot 2\int_{0}^{2}\left(t^2-1\right)t^2dt=4\int_{0}^{2}\left(t^4-t^2\right)t^2dt=
$$

$$
=4\left(\frac{t^5}{5}\Big|_{0}^{2}-\frac{t^3}{3}\Big|_{0}^{2}\right)=4\left(\frac{32}{5}-\frac{8}{3}\right)=4\frac{96-40}{15}=\frac{4\cdot 56}{15}=\frac{224}{15}.
$$

What had to be proved.

11.2. Change of Variables in a Double Integral

The change of a variable in a double integral is applied for the simplicity of calculation corresponding to the way how we did it in the definite integral.

Turning from the Cartesian coordinates x, y to the curve-lined coordinates u, v which are corresponded to the Cartesian ones as $x=x(u,v)$, $y=y(u,v)$, when functions $x(u,v)$, $y(u,v)$ have continuous partial derivatives in the area of the domain D_1 on the plane uO_1v and the determinant Jacobi or Jacobian is not equal to zero: $J=\begin{vmatrix}\frac{\partial x}{\partial U} & \frac{\partial x}{\partial V}\\ \frac{\partial y}{\partial U} & \frac{\partial y}{\partial V}\end{vmatrix}\neq 0$ the formula of the turning can be presented as follows:

$$\iint_D f(x,y)\,dxdy = \iint_{D_1} f\big(x(u,v),y(u,v)\big)|J|\,dudv.$$

Here we can see a single-valied correspondence between the points of the domain D belonging to the plane xoy and the domain D_1 belonging to the plane uO_1v.

In the case of turning into the polar coordinates ρ, φ,

where $\quad x = \rho\cos\varphi$, $y = \rho\sin\varphi$

the change of variables is as follows:

$$\iint_D f(x,y)\,dxdy = \iint_D f(\rho\cos\varphi, \rho\sin\varphi)\,\rho\,d\rho\,d\varphi,$$

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial \rho} & \dfrac{\partial x}{\partial \varphi} \\ \dfrac{\partial y}{\partial \rho} & \dfrac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos\varphi & -\rho\sin\varphi \\ \sin\varphi & \rho\cos\varphi \end{vmatrix} = \rho.$$

The turning into the polar coordinates takes place in the case if integrand is presented as $f\left(x^2+y^2\right)$, and the D is presented as a circle, a ring or their parts. Combining the beginning of the Cartesian and polar systems of coordinates, the domain D (Fig.31) is depicted in the polar coordinates as follows:

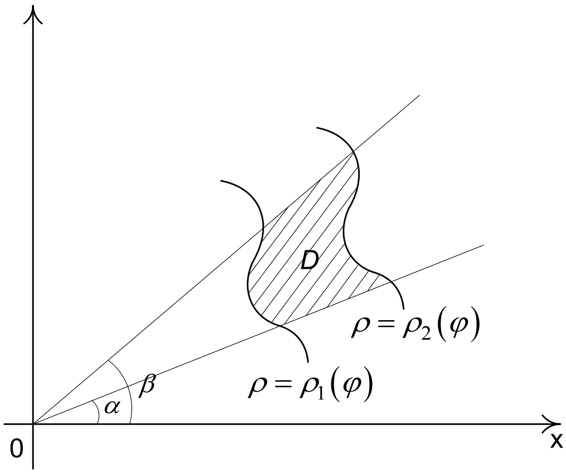


Fig.31

$\alpha < \beta$

$\rho_1(\varphi) \le \rho_2(\varphi)$

$\rho_1(\varphi)$, $\rho_2(\varphi)$ – single-valued functions, when $\alpha \le \varphi \le \beta$.

Double integral is calculated according to the formula

$$\iint_D f(\rho\cos\varphi, \rho\sin\varphi)\,\rho\,d\rho\,d\varphi = \int_\alpha^\beta d\varphi \int_{\rho_1(\varphi)}^{\rho_2(\varphi)} f(\rho\cos\varphi, \rho\sin\varphi)\,\rho\,d\rho =$$

$$=\int_{\alpha}^{\beta}\left[\int_{\rho_1(\varphi)}^{\rho_2(\varphi)} f\left(\rho\cos\varphi,\rho\sin\varphi\right)\rho d\rho\right]d\varphi .$$

Calculating internal integral on ρ, φ is considered to be constant.

Example. Let's calculate a double integral $\iint\limits_D \sqrt{a^2-x^2-y^2}\,dxdy$, where D − is the first quarter of the circle $x^2+y^2\le a^2$.

Solution. Let's turn into the polar coordinates: $x=\rho\cos\varphi$, $y=\rho\sin\varphi$.

$$\iint\limits_D \sqrt{a^2-x^2-y^2}\,dxdy=\iint\limits_D \sqrt{a^2-\rho^2\left(\cos^2\varphi+\sin^2\varphi\right)}\rho d\rho d\varphi=$$

$$=\int_0^{\pi/2}\left[\int_0^a \sqrt{a^2-\rho^2}\,\rho d\rho\right]d\varphi=\frac{1}{2}\int_0^{\pi/2}\left[-\int_0^a\left(a^2-\rho^2\right)^{\frac{1}{2}}d\left(\rho^2+a^2\right)\right]d\varphi=$$

$$=\frac{1}{2}\int_0^{\pi/2}d\varphi\left(-\frac{2\left(a^2-\rho^2\right)^{\frac{3}{2}}}{3}\Bigg|_0^a\right)=-\frac{\pi}{4}\cdot\frac{2}{3}\left(0-a^3\right)=\frac{\pi a^3}{6};$$

11.3. Application of a Double Integral. The area of the figure

$S=\iint\limits_D dxdy$ − is an area of domain D.

If D: $a\le x\le b$; $\varphi_1(x)\le y\le\varphi_2(x)$, then $S=\int_a^b dx\int_{\varphi_1(x)}^{\varphi_2(x)}dy=\int_a^b\left[\varphi_2(x)-\varphi_1(x)\right]dx$.

If domain D is set in the polar coordinates: $\alpha\le\varphi\le\beta$; $\rho_1(\varphi)\le\rho\le\rho_2(\varphi)$.

Then $S=\iint\limits_D \rho d\rho d\varphi=\int_\alpha^\beta d\varphi\int_{\rho_1(\varphi)}^{\rho_2(\varphi)}\rho d\rho=$.

Example. Let's calculate the area of the figure bounded by the lines $y=4x-x^2$; $x+y=4$.

Solution. Let's make a drawing (Fig.32) and find the points of intersection

$$y=4x-x^2=-\left(x^2-4x+4-4\right)=4-(x-2)^2;\ y=4-x$$

$4x-x^2=4-x$;

$x(4-x)-(4-x)=0$;

$(4-x)(x-1)=0$;

$x_1 = 1$; $x_2 = 4$.

$$S = \iint_D dxdy = \int_1^4 dx \int_{4-x}^{4x-x^2} dy = \int_1^4 \left[4x - x^2 - 4 + x\right] dx = \frac{5x^2}{2}\Bigg|_1^4 - \frac{x^3}{3}\Bigg|_1^4 - 4x\Bigg|_1^4 =$$

$$= \frac{5}{2}(16-1) - \frac{1}{3}(64-1) - 4(4-1) = \frac{75}{2} - 21 - 12 = 4{,}5 \text{ (sq. units)}$$

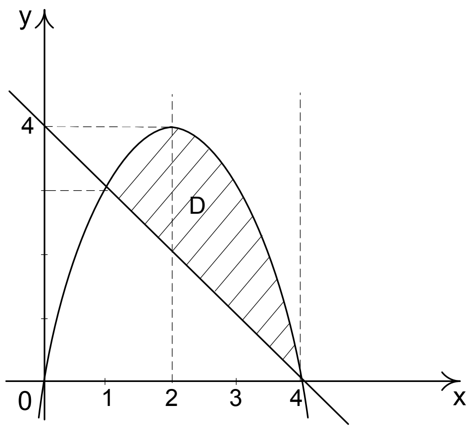


Fig.32

Example. Let's calculate an area of the figures bounded by the lines $\rho = 1$, $\rho = \frac{2}{\sqrt{3}} sin\varphi$ (outside $\rho = 1$).

Solution. Let's draw lines and find the points of intersection. Let's make a drawing (Fig.33) $\rho = 1$ is a circle of the radius and is equal to 1.

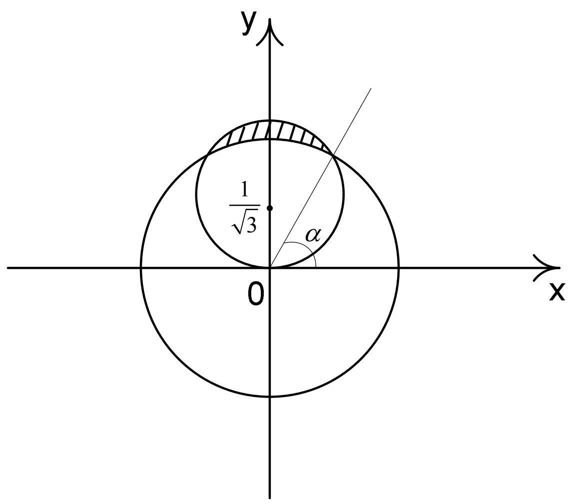


Fig. 33

For drawing $\rho=\frac{2}{\sqrt{3}}sin\varphi$ it's better to use the Cartesian coordinates: $x=\rho cos\varphi$, $y=\rho sin\varphi$, $\rho^2=x^2+y^2$. Then $\sqrt{x^2+y^2}=\frac{2}{\sqrt{3}}\frac{y}{\sqrt{x^2+y^2}}$; $x^2+y^2=\frac{2}{\sqrt{3}}y$;

$x^2+y^2-2\frac{1}{\sqrt{3}}y+\frac{1}{3}-\frac{1}{3}=0$; $x^2+\left(y-\frac{1}{\sqrt{3}}\right)^2=\left(\frac{1}{\sqrt{3}}\right)^2$.

This is a radius circle $\frac{1}{\sqrt{3}}$ with the centre at the point $\left(0;\frac{1}{\sqrt{3}}\right)$. For calculating the angled α,

let's equal $\rho=1$ and $\rho=\frac{2}{\sqrt{3}}sin\varphi$.

$1=\frac{2}{\sqrt{3}}sin\alpha$; $sin\alpha=\frac{\sqrt{3}}{2}$; $\alpha=\frac{\pi}{3}$. Taking into account symmetry of the problem, let's consider change of the angle φ from $\alpha=\frac{\pi}{3}$ to $\beta=\frac{\pi}{2}$.

Then $S=2\int\limits_{\pi/3}^{\pi/2}d\varphi\int\limits_{1}^{\frac{2}{\sqrt{3}}sin\varphi}\rho d\rho=\frac{2}{2}\int\limits_{\pi/3}^{\pi/2}\left[\rho^2\Bigg|_{1}^{\frac{2}{\sqrt{3}}sin\varphi}\right]du=$

$$=\int\limits_{\pi/3}^{\pi/2}\left(\frac{4}{3}sin^2\varphi-1\right)d\varphi=\int\limits_{\pi/3}^{\pi/2}\left(\frac{4}{3}\cdot\frac{1-cos2\varphi}{2}-1\right)d\varphi=$$

$$=\int\limits_{\pi/3}^{\pi/2}\left(\frac{2}{3}-\frac{2-cos2\varphi}{3}-1\right)d\varphi=-\frac{1}{3}\varphi\Bigg|_{\frac{\pi}{3}}^{\frac{\pi}{2}}-\frac{1}{3}sin2\varphi\Bigg|_{\frac{\pi}{3}}^{\frac{\pi}{2}}=$$

$$=-\frac{1}{3}\left(\frac{\pi}{2}-\frac{\pi}{3}\right)-\frac{1}{3}\left(sin\pi-sin\frac{2\pi}{3}\right)=-\frac{\pi}{18}+\frac{\sqrt{3}}{6}=\frac{3\sqrt{3}-\pi}{18} \text{ (sq. units)}$$

Calculation of solid volume

In the case if $z=f(x,y)>0$, the solid volume is calculated according to the formula $V=\iint\limits_D f(x,y)dxdy$, **that corresponds to the geometric meaning of a double integral.**

Example. Calculate a solid volume bounded by the surfaces $4x+2y+z-4=0$; $y=0$; $z=0$.

Solution. Let's calculate the domain of integration. Let $z=0$ (Fig.34) $4x+2y=4$ or $y=-2x+2$. Function $z(x,y)$ is as follows: $z=4-4x-2y$.

Having changed this expression to the following $\frac{x}{1}+\frac{y}{2}+\frac{z}{4}=1$ we can see that this equation of the plane in the intercepts (Fig.35).

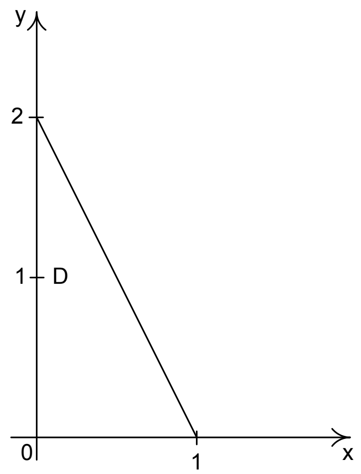


Fig.34

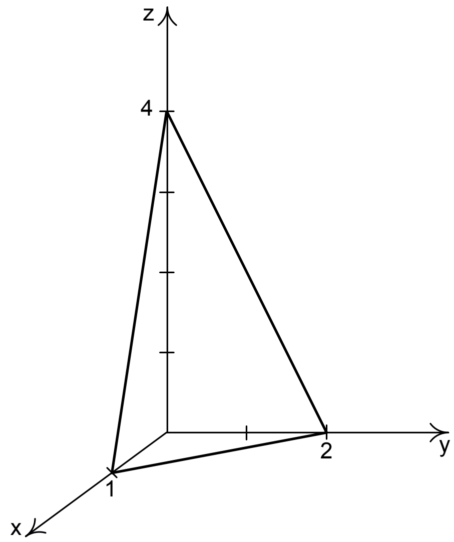


Fig.35

Then $V=\iint_D z(x,y)\,dxdy==\int_0^1 dx\int_0^{2-2x}(4-4x-2y)\,dy=$

$$=\int_0^1\left[4y\Big|_0^{2-2x}-4xy\Big|_0^{2-2x}-\frac{2y^2}{2}\Big|_0^{2-2x}\right]dx=$$

$$=\int_0^1\left(4(2-2x)-4x(2-2x)-2^2(1-x)^2\right)dx=4\int_0^1\left(2-\cancel{2x}-2x+2x^2-\right.$$

$$\left.-1+\cancel{2x}-x^2\right)dx=4\int_0^1\left(1-2x+x^2\right)dx=4\left(x\Big|_0^1-2\frac{x^2}{2}\Big|_0^1+\frac{x^3}{3}\Big|_0^1\right)=\frac{4}{3}\text{ (cub. units).}$$

Let's prove this fact. The volume of the pyramid is $V=\frac{1}{3}S\cdot H=\frac{1}{3}\cdot\frac{2}{2}\cdot4=\frac{4}{3}$ (cub. units).

Example. Calculate a solid volume bounded by the surfaces $x^2+y^2=a^2$; $x^2+z^2=a^2$.

Solution. We will obtain this solid by intersection of two cylindrical surfaces – round cylinders.

Let's draw projections of these surfaces on the plane *xoy* and *xoz* (Fig.36 and Fig.37).

Taking into account the symmetry of the problem we can take one fourth of the circle $x^2 + y^2 = a^2$. As the domain of integration. Taking into account that one half of the volume is above $(z > 0)$ and the second one is below $(z < 0)$ then we are obtaining

$$V = 8\iint_D \sqrt{a^2 - x^2}\,dxdy \text{, Fig.38.}$$

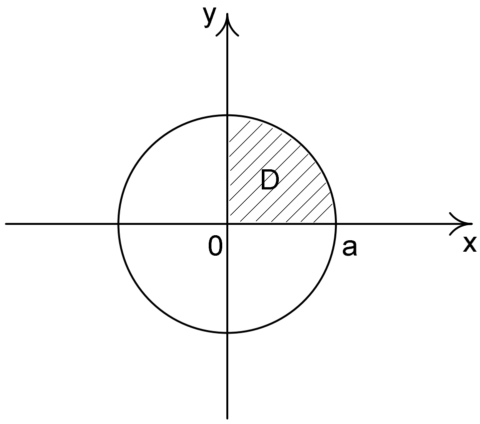


Fig.36

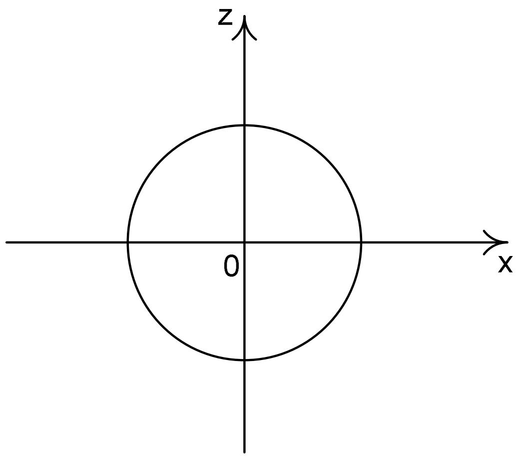


Fig.37

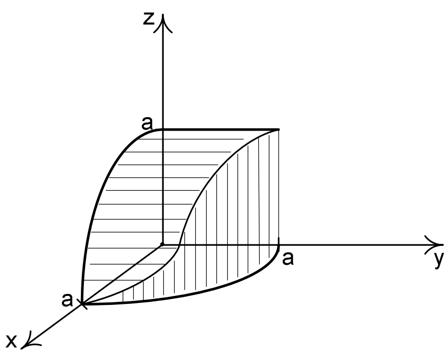


Fig.38

$$V = 8\int_0^a dx \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2}\,dy =$$

$$= 8\int_0^a \left(a^2 - x^2\right)dx = 8\left(a^2 x\Big|_0^a - \frac{x^3}{3}\Big|_0^a \right) =$$

$$= \frac{16}{3}a^3 \text{(cub. units).}$$

The mass of a plane plate with a variable density $\gamma(x, y)$ is calculated according to the formula

$$M = \iint_D \gamma(x, y)\,dxdy \text{, where } D \in xoy.$$

Static moments Mx, My of the plate relatively to the axes ox and oy are calculated according to the formulae

$$Mx = \iint_D y\gamma(x,y)\,dxdy$$

$$My = \iint_D x\gamma(x,y)\,dxdy$$

If a plate is homogenous, then $\gamma = const$.

Inertia moments J_x, J_y of a plate relatively to the axes ox and oy are equal to

$$J_x = \iint_D y^2\gamma(x,y)\,dxdy;\quad J_y = \iint_D x^2\gamma(x,y)\,dxdy$$

The inertia moments J_0 of a plate relatively to the beginning of the coordinates − a polar inertia moment is equal to $J = \iint_D (x^2 + y^2)\gamma(x,y)\,dxdy$.

Let's notice that $J_0 = J_x + J_y$.

We will obtain geometric inertia moment when $\gamma = 1$. Gravity center coordinates of a plate are equal to

$$\bar{x} = \frac{My}{M};\ \bar{y} = \frac{Mx}{M}$$

If $\gamma = const$, then $\bar{x} = \dfrac{\iint_D xdxdy}{\iint_D dxdy} = \dfrac{\iint_D xdxdy}{S}$; $\bar{y} = \dfrac{\iint_D ydxdy}{\iint_D dxdy} = \dfrac{\iint_D ydxdy}{S}$.

Home task

Put the limits of integration according to the formulae I and II for domain D.

1. $D:$ $x^2 = 2 - y$; $x - y = 0$.
2. $D:$ $x = \sqrt{4 - y^2}$; $y \ge 0$; $y = 1$.
3. $D:$ $y = x$; $y = -x + 2$; $y \ge 0$.

Calculate double integrals:

4. $\iint_D (y - x)\,dxdy$, $D:$ $x + y = 0$; $y = x^2$.
5. $\iint_D e^y\,dxdy$, $D:$ $y = \ln x$; $y = 0$; $x = 2$.
6. $\iint_D e^{x^2+y^2}\,dxdy$, $D:$ $x \ge 0$; $y \ge 0$; $x^2 + y^2 = 1$.

7. $\int_0^3 dx \int_0^{\sqrt{9-x^2}} \frac{dy}{x^2+y^2+1}$.

Calculate the area of a plane figure bounded by the lines:

8. $x^2+y^2=4$; $y \le x$; $y \ge 0$.

9. $y = \sin x$; $y \le -x+\frac{\pi}{2}+1$; $y \ge 0$.

10. $\rho = a\sqrt{\sin 2\varphi}$.

11. $\left(x^2+y^2\right)^2 = a^2xy$.

Calculate the volume of the solid bounded by the pointed surfaces:

12. $y=1-x^2$; $x+y+z-3=0$; $x \ge 0$; $y \ge 0$.

13. $z=x$; $y=4$; $x^2+y^2=25$; $x \ge 0$; $y \ge 0$; $z \ge 0$.

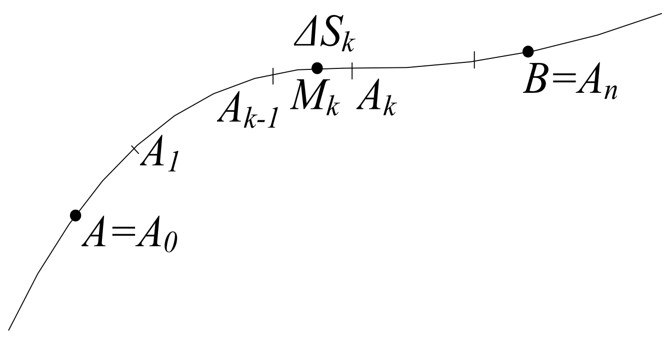
14. $x^2+(y-2)^2=4$; $z^2=4-y$; $z \ge 0$.

12. CONTOUR INTEGRALS

12.1. Contour Integrals of the 1st kind

Contour integral of the 1st kind or contour integral on the arc *AB* of a function $f(x,y)$, a definite and continuous smooth curve *K* from *A* to *B* at all points is a limit of integral sum $\sum_{k=1}^{n} f(\xi_k,\eta_k)\Delta S_k$ under the condition that $\max \Delta S_k \to 0$

$$\int_{AB} f(x,y)dS = \lim_{\max \Delta S \to 0} \sum_{k=1}^{n} f(\xi_k,\eta_k)\Delta S_k \text{ (Fig.39).}$$



$M_k(\xi_k,\eta_k)$ dS – arc differential

Fig.39

If the equation of the curve *K* is represented as $y=\varphi(x)$ ($a\le x\le b$), then contour integral of the 1st kind is calculated according to the formula

$$\int\limits_{AB} f(x,y)dS=\int\limits_a^b f[x,\varphi(x)]\sqrt{1+[\varphi'(x)]^2}\,dx.$$

If the curve *K* is given by the following parametric equations $x=x(t),\ y=y(t)$, where $t_1\le t\le t_2$, then

$$\int\limits_K f(x,y)dS=\int\limits_{t_1}^{t_2} f[x(t),y(t)]\sqrt{[x'(t)]^2+[y'(x)]^2}\,dt.$$

If the curve *K* is given in the space $x=x(t),\ y=y(t),\ z=z(t)$, $t_1\le t\le t_2$, then

$$\int\limits_K f(x,y,z)dS=\int\limits_{t_1}^{t_2} f[x(t),y(t),z(t)]\sqrt{[x'(t)]^2+[y'(x)]^2+[z'(x)]^2}\,dt.$$

If $f(x,y)>0$, then phisycal meaning of the contour integral of the 1st kind – is a weight of the curve *K*, with variable density $\gamma=f(x,y)$.

Basic properties of the contour integral of the first kind

1. $\int\limits_{AB} f(x,y)dS=\int\limits_{BA} f(x,y)dS$
2. $\int\limits_K [f_1(x,y)\pm f_2(x,y)]dS=\int\limits_K f_1(x,y)dS\pm\int\limits_K f_2(x,y)dS$
3. $\int\limits_K \lambda f\,(x,y)dS=\lambda\int\limits_K f\,(x,y)dS$, $\quad\lambda=const$
4. If $K=K_1\bigcup K_2$, then $\int\limits_K f\,(x,y)dS=\int\limits_{K_1} f\,(x,y)dS+\int\limits_{K_2} f\,(x,y)dS$.

If the curve K is defined in the polar coordinates by the equation $\rho=\rho(\varphi)$, $\alpha\le\varphi\le\beta$, then

$$\int\limits_K f\,(x,y)dS=\int\limits_\alpha^\beta f(\rho\cos\varphi,\rho\sin\varphi)\sqrt{[\rho(\varphi)]^2+[\rho'(\varphi)]^2}\,d\varphi.$$

Contour integral on the length of an arc has a wide application to the problems in mathematics and mechanics.

The length of the curve *AB*: $L=\int\limits_{AB} dS$.

The weight of a material curve with a variable density $\gamma=f(x,y)$: $M=\int\limits_{AB} f(x,y)dS$.

Static moments of a material curve AB relatively to the axes OX and OY and its gravity centre coordinates $\overline{X}, \overline{Y}$ are calculated according to the formulae

$$S_x = \int\limits_{AB} y\ f(x,y)dS; \quad S_y = \int\limits_{AB} x\ f(x,y)dS;$$

$$\overline{X} = \frac{S_y}{M}; \quad \overline{Y} = \frac{S_x}{M}.$$

If a curve is homogenous, then

$$\overline{x} = \frac{\int\limits_{AB} xdS}{L}; \quad \overline{y} = \frac{\int\limits_{AB} ydS}{L}.$$

Inertia moments of a material curve AB relatively to the axes OX, OY and the beginning of the coordinates is calculated according to the formulae:

$$I_x = \int\limits_{AB} y^2 f(x,y)dS; \ I_y = \int\limits_{AB} x^2 f(x,y)dS; \ I_0 = \int\limits_{AB} (x^2+y^2) f(x,y)dS.$$

Example. To calculate $\int\limits_K (x+y)dS$, where K – is a straight line from $A(1;2)$ to $B(3;4)$.

Solution. Let's form an equation of the straight line

$$\frac{y-y_A}{y_B-y_A} = \frac{x-x_A}{x_B-x_A}; \quad \frac{y-2}{4-2} = \frac{x-1}{3-1}; \quad y = x+1; \quad y' = 1.$$

$$\int\limits_K (x+y)dS = \int\limits_1^3 (x+x+1)\sqrt{1+1}dx = \sqrt{2}\int\limits_1^3 (2x+1)dx =$$

$$= \sqrt{2}\left(\left.\frac{2x^2}{2}\right|_1^3 + x\Big|_1^3 \right) = \sqrt{2}(9-1+3-1) = 10\sqrt{2}.$$

Example. To calculate the length of the first turn of the circular helix $x = \cos t, \ y = \sin t, \ z = t$.

Solution. $L = \int\limits_K dS = \int\limits_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t + 1}dt = \sqrt{2}\,t\Big|_0^{2\pi} = 2\sqrt{2}\pi$ (length unit).

Example. To calculate gravity centre coordinates of the homogenous arc $y = \sqrt{a^2 - x^2}$.

Solution. The given curve is the upper half of the circle $x^2 + y^2 = a^2$ (Fig.40). Using symmetry of the problem $\overline{x} = 0$.

$$\bar{y} = \frac{\int_{-a}^{a} y\,dS}{L} = \frac{\int_{-a}^{a} y\sqrt{1+\left(\frac{-2x}{2\sqrt{a^2-x^2}}\right)^2}\,dx}{\pi\, a} =$$

$$= \frac{1}{\pi a}\int_{-a}^{a}\sqrt{a^2-x^2}\sqrt{\frac{a^2-x^2+x^2}{a^2-x^2}}\,dx = \frac{1}{\pi}x\Big|_{-a}^{a} = \frac{2a}{\pi}.$$

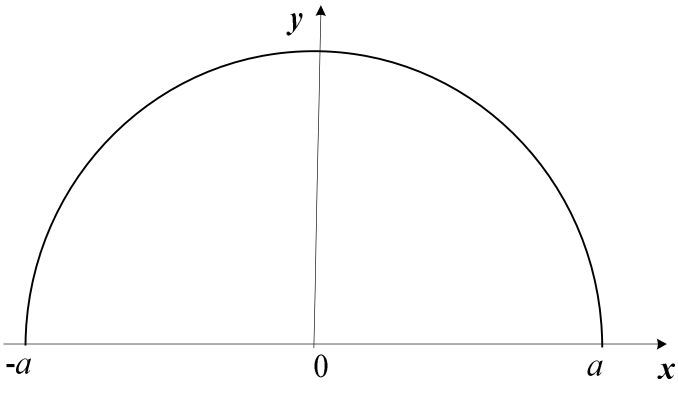


Fig.40

Example. Let's calculate a mass of the lemniscate lobe $\rho = \sqrt{\sin 2\varphi}$ located in the first quadrant if the density is $\gamma(x, y) = x + y$ (Fig.41).

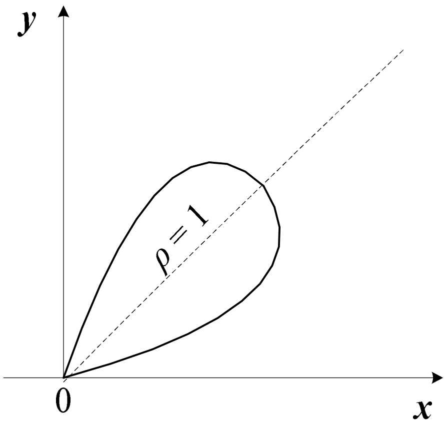


Fig.41

Solution. $M = \int_K (x+y)\,dS$

$$dS = \sqrt{[\rho(\varphi)]^2 + [\rho'(\varphi)]^2}\,d\varphi =$$

$$= \sqrt{\sin 2\varphi + \left(\frac{2\cos 2\varphi}{2\sqrt{\sin 2\varphi}}\right)^2}\,d\varphi =$$

$$= \sqrt{\frac{\sin^2 2\varphi + \cos^2 2\varphi}{\sin 2\varphi}}\,d\varphi = \frac{1}{\rho}\,d\varphi.$$

Then $M = \int_0^{\pi/2} (\rho\cos\varphi + \rho\sin\varphi)\frac{1}{\rho}\,d\varphi = \int_0^{\pi/2} (\sin\varphi + \cos\varphi)\,d\varphi =$

$$= -\cos\varphi\Big|_0^{\pi/2} + \sin\varphi\Big|_0^{\pi/2} = 1 + 1 = 2.$$

12.2. Contour integrals of the 2nd kind

Contour integral of the 2nd kind or contour integral relatively to the coordinates from the $P(x,y)dx+Q(x,y)dy$ on the directed arc AB is a limit of integral sum under the condition $\max\Delta x_k \to 0$ and $\max\Delta y_k \to 0$:

$$\int\limits_{AB} P(x,y)dx+Q(x,y)dy = \lim_{\substack{\max\Delta x_k\to 0\\ \max\Delta y_k\to 0}} \sum_{k=1}^{n}\left[P(\xi_k,\eta_k)\Delta x_k + Q(\xi_k,\eta_k)\Delta y_k\right],$$ where $P(x,y)$ and $Q(x,y)$ are continuous at the points of the AB arc of the smooth curve K: $y=\varphi(x)$ ($a\le x\le b$), Δx_k and Δy_k – are projections of the elementary arc on the axes Ox and Oy (Fig.42).

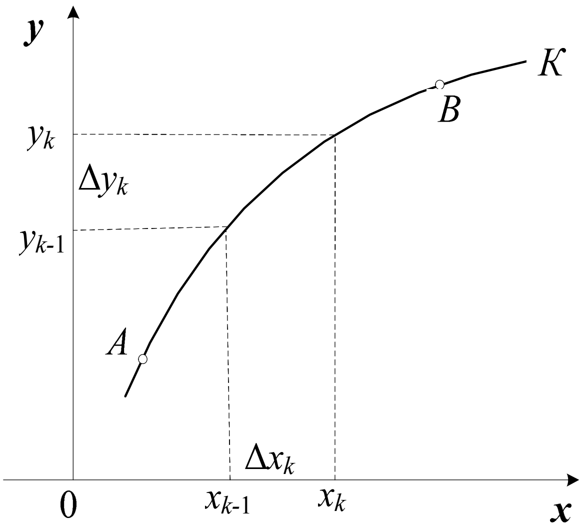


Fig.42

Physical meaning of the contour integral of the 2nd kind is work of a variable force $\vec{F}=P(x,y)\vec{i}+Q(x,y)\vec{j}$ on the curved way from A to B.

Basic properties of the contour integral of the 2nd kind

1. $\int\limits_{BA} Pdx+Qdy = -\int\limits_{AB} Pdx+Qdy$
2. $\int\limits_{AB} Pdx+Qdy = \int\limits_{AB} Pdx + \int\limits_{AB} Qdy$
3. Contour integral on the closed contour (is marked by the symbol $\oint$) doesn't depend on the choice of initial point and depends on the direction of detour curve .

The rest properties are equal to the properties of contour integral of the 1st kind.

Rules of calculation

If a curve *K* is given by the equation $y=\varphi(x)$, $a\le x\le b$, then

$$\int_K P(x,y)dx+Q(x,y)dy=\int_a^b\left[P(x,\varphi(x))+\varphi'(x)Q(x,\varphi(x))\right]dx.$$

If a curve *K* is given parametrical $x=x(t),\ y=y(t)$, $t_1\le t\le t_2$, then

$$\int_K P(x,y)dx+Q(x,y)dy=\int_{t_1}^{t_2}\left[P(x(t),y(t))x'(t)+Q(x(t),y(t))y'(t)\right]dt.$$

If *K* is a space curve $x=x(t),\ y=y(t),\ z=z(t)$, $t_1\le t\le t_2$, then

$$\int_K P(x,y,z)dx+Q(x,y,z)dy+R(x,y,z)dz=$$

$$=\int_{t_1}^{t_2}\left\{P[x(t),y(t),z(t)]x'(t)+Q[x(t),y(t),z(t)]y'(t)+R[x(t),y(t),z(t)]z'(t)\right\}dt.$$

Example. To calculate $\int_{AB}(x-y)^2dx+(x+y)^2dy$, $A(0;1);\ B(3;4)$.

Solution. Let's make the equation of the line *AB*.

$$\frac{y-y_A}{y_B-y_A}=\frac{x-x_A}{x_B-x_A};\quad \frac{y-1}{4-1}=\frac{x-0}{3-0};\quad y=x+1;\quad dy=dx.$$

Let's put *y* and *dy* into the searched integral

$$\int_{AB}(x-y)^2dx+(x+y)^2dy=\int_0^3[(x-x-1)^2+(x+x+1)^2]dx=$$

$$=\int_0^3(1+4x^2+4x+1)dx=2\int_0^3(1+2x+2x^2)dx=$$

$$=2\left[x\Big|_0^3+2\frac{x^2}{2}\Big|_0^3+2\frac{x^3}{3}\Big|_0^3\right]=2[3+9+18]=60.$$

Example. To calculate $\int_{AB}y^2dx+z^2dy+x^2dz$, $A(1;0;2);\ B(3;1;4)$.

Solution. Let's make an equation of the space line

$$\frac{x-1}{3-1}=\frac{y-0}{1-0}=\frac{z-2}{4-2};\quad \frac{x-1}{2}=\frac{y}{1}=\frac{z-2}{2}$$

or in the parametric form

$x-1=2t;\ \ y=t;\ \ z-2=2t;$

$x=2t-1;\ \ y=t;\ \ z=2t+2;\qquad dx=2dt;\ \ dy=dt;\ \ dz=2dt.$

For the point A $t_1 = 0$. For the point B $t_2 = 1$.

Then $\int\limits_{AB} y^2dx + z^2dy + x^2dz = \int\limits_0^1 \left[t^2 \cdot 2 + 2^2(t+1)^2 + (2t+1)^2 \cdot 2\right]dt =$

$= 2\int\limits_0^1 [t^2 + 2t^2 + 4t + 2 + 4t^2 + 4t + 1]dt = 2\int\limits_0^1 [7t^2 + 8t + 3]dt =$

$2\left[7\frac{t^3}{3} + 8\frac{t^2}{2} + 3t\right]\Bigg|_0^1 = 2\left[\frac{7}{3} + 4 + 3\right] = 2 \cdot 7\left[\frac{1}{3} + 1\right] = \frac{56}{3}.$

Example. To calculate $\int\limits_K x^2ydx + x^3dy$, where K is a contour, bounded by parabolies $y^2 = x,\ x^2 = y$ and passing counter – clockwise.

Solution. Let's make a drawing (Fig.43).

$\int\limits_K x^2ydx + x^3dy = \int\limits_{OAB} x^2ydx + x^3dy + \int\limits_{BCO} x^2ydx + x^3dy$.

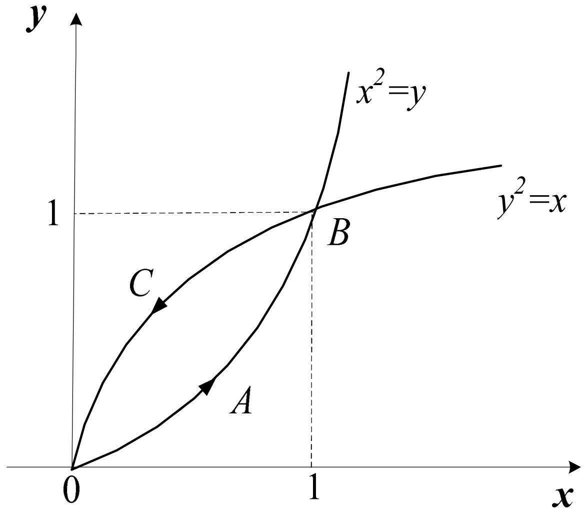


Fig.43

For contour OAB:

$y = x^2;\ dy = 2xdx$;

x changes from 0 to 1.

For contour BCO: $y = \sqrt{x};\ dy = \dfrac{dx}{2\sqrt{x}}$;

x changes from 1 to 0.

Then $J = \int\limits_0^1 (x^2 \cdot x^2 + x^3 \cdot 2x)dx + \int\limits_1^0 (x^2\sqrt{x} + x^3\frac{1}{2\sqrt{x}})dx =$

$= 3\int\limits_0^1 x^4dx + \int\limits_1^0 (x^{5/2}\frac{1}{2}x^{5/2})dx = \frac{3}{5}x^5\Big|_0^1 + \frac{3}{2}\frac{2x^{7/2}}{7}\Bigg|_1^0 = \frac{3}{5} + \frac{3}{7}(-1) = \frac{3(7-5)}{35} = \frac{6}{35}.$

12.3. Independence of the contour integral of the 2nd kind from the way of integration. Calculation of the function of its full differential

For the contour integral $\int_K P(x,y)dx + Q(x,y)dy$ not to be dependent on the way of integration in the single-connected domain *D*, where functions $P(x,y)$ and $Q(x,y)$ are continuous together with their partial derivatives it's necessary and sufficient that in each point of this domain the condition $\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x}$ must take place, contour *K* lies in the domain *D* completely.

If these conditions are followed , the contour integral at any closed contour *C* lying in the domain *D*, is equal to 0

$$\oint_C P(x,y)dx + Q(x,y)dy = 0.$$

While calculating the integral not depending on the way of integration is the best way one must take a broken line, links of which are parallel to the axes of coordinates.

If these conditions $\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x}$ are followed by the integrant expression $P(x,y)dx + Q(x,y)dy$ is a total differential of some unique function $U = U(x,y)$, i.d. $dU(x,y) = P(x,y)dx + Q(x,y)dy$.

Function $U(x,y)$ is called an anti-derivative and is calculated by integration from some fixed point $A(x_0, y_0)$ to the variable point *B*(*x,y*), i.d.

$$U(x,y) = \int_{x_0}^{x} P(x,y_0)dx + \int_{y_0}^{y} Q(x,y)dy + C \quad \text{or}$$

$$U(x,y) = \int_{y_0}^{y} Q(x_0,y)dy + \int_{x_0}^{x} P(x,y)dx + C.$$

In the capacity of the fixed point it is necessary to take a point (0;0), if functions $P(x,y)$ and $Q(x,y)$ stay continuous.

Example. To check if integrant expression is a total differential of some function $U(x,y)$ and in case if the answer is positive to find this function

$$\int_K (x^2 - 2xy^2 + 1)dx + (y^2 - 2x^2y - 2)dy.$$

Solution. Let's check the condition $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

$$\frac{\partial}{\partial y}(x^2 - 2xy^2 + 1) = -4xy; \quad \frac{\partial}{\partial x}(y^2 - 2x^2y - 2) = -4xy.$$

The condition takes place for all x and y lying on the domain xOy, means that integrant expression is a total differential.

Let's find an anti-derivative

$$U(x,y) = \int_{(x_0,y_0)}^{(x,y)} (x^2 - 2xy^2 + 1)dx + (y^2 - 2x^2y - 2)dy.$$

As the given integral doesn't depend on the contour of integration let's take a broken line OAB in the capacity of the contour (Fig.44).

On the segment OA: $y = 0$; $dy = 0$; x changes from 0 to x.

On the segment AB: $x = x$; dx=0; y changes from 0 to y.

Then $U(x,y) = \int\limits_{OAB} = \int\limits_{OA} + \int\limits_{AB} = \int_0^x (x^2 - 2x \cdot 0 + 1)dx + \int_0^y (y^2 - 2x^2y - 2)dx =$

$$= \frac{x^3}{3}\Big|_0^x + x\Big|_0^x + \frac{y^3}{3}\Big|_0^y - 2x^2\frac{y^2}{2}\Big|_0^y - 2y\Big|_0^y = \frac{x^3}{3} + x + \frac{y^3}{3} - x^2y^2 - 2y + C.$$

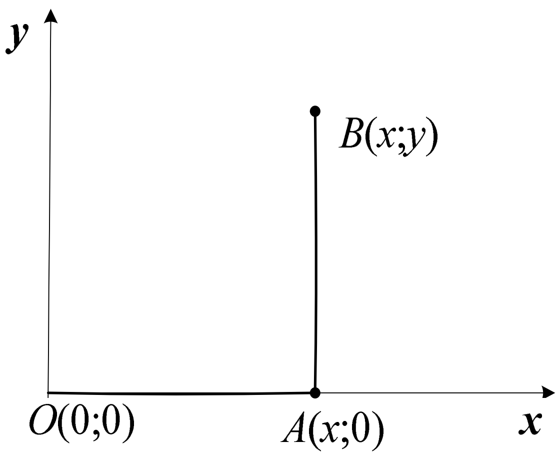


Fig.44

Let's check:

$$P(x,y)=\frac{\partial U(x,y)}{\partial x}=x^2+1-2xy^2; \qquad Q(x,y)=\frac{\partial U(x,y)}{\partial y}=y^2-2yx^2-2,$$

corresponds to the condition.

Example. To calculate $\oint\limits_K xdy+ydx$ along the closed contours:

1) $y=x^3$; $y=1$; $x=0$ (Fig.45);

2) $x=\cos t$; $y=\sin t$ (Fig.46).

1) $\oint\limits_K = \int\limits_{OA} + \int\limits_{AB} + \int\limits_{BO} = \int\limits_0^1 (x\cdot 3x^2+x^3)dx + \int\limits_1^0 (x\cdot 0+1)dx + = \int\limits_1^0 (0dy+y\cdot 0)dx =$

$= 4\frac{x^4}{4}\Big|_0^1 + x\Big|_1^0 = 1-1=0$, that corresponds to the theory, as $P(x,y)=y$; $Q(x,y)=x$;

$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}=1;$

2) $\oint\limits_K ydx+xdy = \int\limits_0^{2\pi}(\sin t(-\sin t)+\cos t\cdot\cos t)dt = \int\limits_0^{2\pi}\cos 2t dt =$

$=\frac{1}{2}\sin 2t\Big|_0^{2\pi} = \frac{1}{2}(\sin 4\pi - \sin 0)=0.$

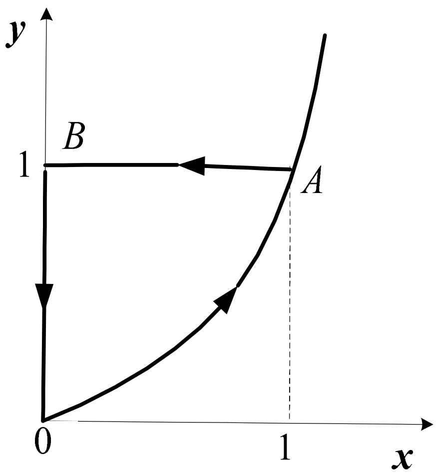


Fig.45

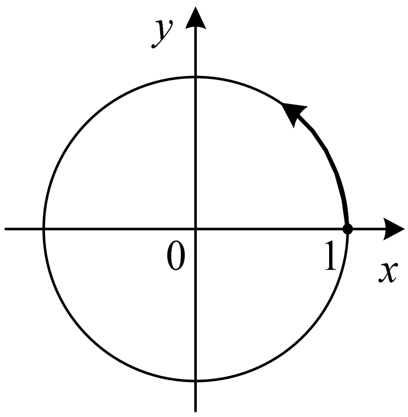


Fig.46

Example. Calculate the work of variable force $\vec{F}=3x^4\vec{i}+xy\vec{j}$ along the curve $y=x^2$ from point $O(0;0)$ to point $A(1;1)$.

Solution. The work of variable force is a contour integral of the 2[nd] kind along the contour *OA*

$$A = \int\limits_{OA} 3x^4 dx + xydy = = \int\limits_0^1 (3x^4 + x \cdot x^2 \cdot 2x)dx + \int\limits_1^0 (x \cdot 0 + 1)dx = 5\int\limits_0^1 x^4 dx = \frac{5}{5}x^5\Big|_0^1 = 1.$$

12.4. Green's formula

Let functions $P(x, y)$ and $Q(x, y)$ are continuous together with their partial derivatives $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ in the closed domain *D* including its border *C*, then Green's formula takes place

$$\oint\limits_C Pdx + Qdy = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy.$$

Detour of the closed contour *C* takes place in the positive direction, i.d. the domain *D* remains on the left.

Example. With the help of Green's formula let's calculate $\oint\limits_C -x^2 ydx + xy^2 dy$, where a contour *C* is a circle $x^2 + y^2 = a^2$.

Solution. $P(x, y) = -x^2 y$; $Q(x, y) = xy^2$.

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y^2 + x^2$, then $\oint\limits_C -x^2 ydx + xy^2 dy = \iint\limits_D \left(x^2 + y^2 \right) dxdy$.

Moving to the polar coordinates $x = \rho\cos\varphi$; $y = \rho\sin\varphi$, $0 \le \varphi \le 2\pi$, we have

$$\iint\limits_D \left(x^2 + y^2 \right) dxdy = \iint\limits_D \rho^2 \rho d\rho d\varphi = \int\limits_0^{2\pi} d\varphi \int\limits_0^a \rho^3 d\rho = \frac{2\pi a^4}{4} = \frac{\pi a^4}{2}.$$

Having put $P(x, y)$=0; $Q(x, y)$=x into the Green's formula we get $\iint\limits_D (1 - 0)dxdy = \oint\limits_C 0dx + xdy$,

then $P(x, y) = -y$; $Q(x, y)$=0, we get $\iint\limits_D (0 + 1)dxdy = \oint\limits_C -ydx + 0dy$.

Added these two equalities and we have $2S = \oint\limits_C xdy - ydx$ or $S = \frac{1}{2}\oint\limits_C xdy - ydx$,

at detour along the contour *C*, the domain *D* remains on the left.

Example. To calculate the square of the figure bounded by the lines $y = \frac{1}{x}$

and $y=-x+\frac{5}{2}$ (Fig.47).

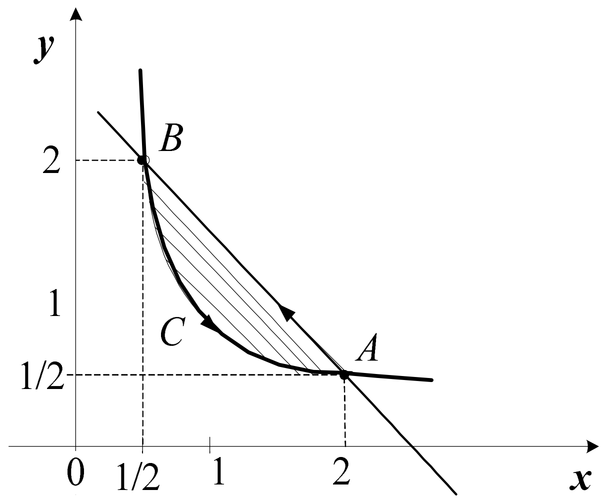


Fig.47

Solution. Let's find intersection points of the lines.

$$\frac{1}{x}=-x+\frac{5}{2};\ \ 1=-x^2+\frac{5}{2}x;\ \ x^2-\frac{5}{2}x+1=0;\ \ x_{1,2}=\frac{\frac{5}{2}\pm\sqrt{\frac{25}{4}-4}}{2}=\frac{\frac{5}{2}\pm\frac{3}{2}}{2}=\begin{cases}2\\ \frac{1}{2}\end{cases}.$$

$$S=\frac{1}{2}\oint_C xdy-ydx=\int_{AB}+\int_{BCA}=\frac{1}{2}\int_2^{1/2}\left[x(-1)+\left(x-\frac{5}{2}\right)\right]dx+$$

$$+\frac{1}{2}\int_{1/2}^{2}\left[x\left(-\frac{1}{x^2}\right)-\frac{1}{x}\right]dx=-\frac{5}{4}x\Big|_2^{1/2}-\frac{2}{2}\int_{1/2}^{2}\frac{1}{x}dx=-\frac{5}{4}\left(\frac{1}{2}-2\right)-\ln x\Big|_{1/2}^{2}=$$

$$=\frac{5}{4}\cdot\frac{3}{2}-\ln 2+\ln\frac{1}{2}=\frac{15}{8}-2\ln 2=\frac{15}{8}-\ln 4 \text{ (square units).}$$

Home task

To calculate the following contour integrals

1. $\oint\limits_{L_{OA}}\frac{dS}{\sqrt{8-x^2-y^2}}$, where L_{OA} is a straight line, $O(0;0)$: $A(2;2)$.

2. $\oint\limits_{L} xydS$, where L: $OABCD$, $O(0;0)$; $A(4;0)$; $B(4;2)$; $C(0;2)$.

3. $\oint\limits_{L}(x^2+y^2+z^2)dS$, where L: $x=\cos t;\ y=\sin t;\ z=\sqrt{3}t,\ 0\le t\le 2\pi$.

4. $\oint\limits_L arctg\frac{y}{x}dS$, where L: $\rho = 2(1+\cos\varphi)$, $0 \le \varphi \le \frac{\pi}{2}$.

5. $\oint\limits_{L_{OA}} (xy - y^2)dx + xdy$, where L_{OA}: $y = 2x^2$; $O(0;0)$; $A(1;2)$.

6. $\oint\limits_L \frac{x}{y}dx + \frac{1}{y-2}dy$, where L: $x = 2(t - \sin t)$; $y = 2(1-\cos t)$; $\frac{\pi}{6} \le t \le \frac{\pi}{3}$.

7. $\oint\limits_{L_{OA}} (xy - y^2)dx + xdy$, where L_{OA}: $y = 2\sqrt{x}$ $O(0;0)$: $A(1;2)$.

8. $\oint\limits_{L_{AB}} \cos z dx - \sin x dz$, where L_{AB} is a straight line, $A(2;0;-2)$: $B(-2;0;2)$.

To find $U(x, y)$, if

9. $dU(x, y) = (2x - 3y^2 + 5)dx + (1 - 6xy)dy$.

10. $dU(x, y) = (y^2 e^{xy} - 2)dx + (1 + xy)e^{xy}dy$.

To calculate the following integrals by means of the Green's formula

11. $\oint\limits_C \sqrt{x^2+y^2}dx + \left[xy + \ln(x + \sqrt{x^2+y^2}\right]ydy$, where contour C: ;

$0 \le y \le 2$.

12. $\oint\limits_C 2(x^2+y^2)dx + (x+y)^2 dy$, where K: ΔABC, $A(1;1)$; $B(2;2)$: $C(1;3)$.

13. THE SERIES

13.1. Numerical Series

Let's consider infinite sequence of numbers $u_1, u_2, u_3, ..., u_n, ...,$ where $u_n = f(n)$.

The expression $u_1 + u_2 + u_3 + ... + u_n + ...,$ is called the infinite numerical series.

$u_1, u_2, u_3, ...$ are terms of the series, u_n is the n-th term of the series.

If $u_n > 0$ then the series is called a numerical series with *nonnegative* members.

$$\sum_{n=1}^{\infty} u_n = \underbrace{u_1 + u_2 + u_3 + ... + u_n}_{S_n \text{ – the } n^{\text{th}} \text{ partial sums}} + \underbrace{u_{n+1} + u_{n+2} + ...}_{R_n \text{ – the } n^{\text{th}} \text{ remainder}}$$

The series is called convergent, if there exists some finite limit $\lim_{n\to\infty} S_n = S$ − is sum of the series, but if there is not such a limit then the series is called divergent.

The series composed by the members of decreasing geometric progression

$$a + aq + aq^2 + ... + aq^n + ..., (|q| < 1)$$

converges and has the sum $\frac{a}{1-q}$.

$1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} + ...$ the harmonic series – diverges.

The generalized harmonic series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} p > 1, \text{ converges} \\ p \le 1, \text{diverges} \end{cases}$

The necessary sign of the convergence of the series: if the series $\sum_{n=1}^{\infty} u_n$ converges then $\lim_{n\to\infty} u_n = 0$ and vice verse the sufficient sign of the divergence, − if the $\lim_{n\to\infty} u_n \ne 0$, then the series diverges.

The basic theorem of the converging numeric series.

Theorem 1. If the series $u_1 + u_2 + u_3 + ... + u_n$ converges then the series $u_{n+1} + u_{n+2} + u_{n+3} + ...$, converges too, obtained from the given by the dropping of the first n − members and vice versa on the contrary from the convergence of R_n remainder of the series the convergence of the given series follows.

Theorem 2. If the series $u_1 + u_2 + ... + u_n +$ converges and its sum is equal to S, then the series $au_1 + au_2 + au_3 + ...$ converges as well and its sum is equal to aS.

Theorem 3. If series $u_1 + u_2 + u_3 + ...$ and $v_1 + v_2 + v_3 + ...$ converges and have sums S and σ, then the series $(u_1 + v_1) + (u_2 + v_2) + (u_3 + v_3) + ...$ converges and its sum is equal to $S + \sigma$.

The basic signs of numeric series convergence with nonnegative members.

The 1st sign of comparison.

Let two series $u_1 + u_2 + u_3 + ... + u_n + ...$ and $v_1 + v_2 + v_3 + ... + v_n + ...$ are given and $u_n \le v_n$ (let even $u_n < v_n$ not for all n but beginning with some $n = N$). Then from the convergence of the series $\sum_{n=1}^{\infty} v_n$ the convergence of the series $\sum_{n=1}^{\infty} u_n$

follows and vice versa, from the divergence of the series $\sum_{n=1}^{\infty} u_n$ the divergence of the series $\sum_{n=1}^{\infty} v_n$ follows.

The 2nd sign of comparison. If there is a finite limit not equal to zero $\lim_{n\to\infty}\frac{u_n}{v_n}$ or $\lim_{n\to\infty}\frac{v_n}{u_n}$ then both of the series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ have the same, i.d. they converge or diverge simultaneously.

<u>The sign by d'Alembert</u>. If there is a limit $D=\lim_{n\to\infty}\frac{u_{n+1}}{u_n}$, for the series $\sum_{n=1}^{\infty} u_n$ then

$$D=\begin{cases} <1, \text{ the series converges} \\ >1, \text{ the series diverges} \\ =1, \text{ the sign by d'Alambert doesn't correspond} \\ \quad \text{to the question of the series convergence} \end{cases}$$

<u>The radical sign by Cauchy</u>.

If there is a limit $\sqrt{k}=\lim_{n\to\infty}\sqrt[n]{u_n}$ for the series $\sum_{n=1}^{\infty} u_n$ then

$$\sqrt{k}=\begin{cases} <1, \textit{ the series converges} \\ >1, \textit{ the series diverges} \\ =1, \textit{ the radical sign by Cauchy doesn't correspond} \\ \quad \textit{to the question of the series convergence} \end{cases}$$

<u>The Integral Sign by Cauchy</u>.

If $f(x)$ at $x\ge 1$ is a continuous nonnegative monotonically decreasing function then the series $\sum_{n=1}^{\infty} u_n$, where $u_n=f(n)$, converges or diverges simultaneously with the convergence and divergence of the improper integral $\int_1^{\infty} f(x)\,dx$.

The alternating in sign series is a series of the kind $u_1-u_2+u_3-u_4+\ldots$ where $u_n>0$, i.d. the series of two neighbouring members which have different signs.

<u>The sign by Leibnitz</u>. The alternating in sign series converges if

1) modules of the members discrease $u_1 > u_2 > u_3 > ...$

2) the common member tends to zero: $\lim\limits_{n\to\infty} u_n = 0$.

$$S_n = u_1 - u_2 + u_3 - u_4 + ... + (-1)^{n-1} u_n$$

$$R_n = S - S_n = (-1)^n (u_{n+1} - u_{n+2} + u_{n+3} - u_{n+4} + ...)$$

$$|R_n| < u_{n+1}$$

The alternating series are such series with terms which are real numbers of any sign.

The alternating series in sign are proper case of the alternating series.

The alternating series $u_1 + u_2 + u_3 + ...$ converges if the series $|u_1| + |u_2| + |u_3| + ...$ converges.

In this case the initial series $\sum\limits_{n=1}^{\infty} u_n$ is called absolutely converging.

The converging series $\sum\limits_{n=1}^{\infty} u_n$ is called conditionally converging if the series $|u_1| + |u_2| + |u_3| + ...$ diverges.

In absolutely converging series you can make a rearrangement of the infinite number of members convergence and a sum of series will not change.

This statement is not right for conditionally converging series.

For finding the common series member it is useful to remember the expression for the common member of arithmetic progression with a difference d

$$a_n = a_1 + d(n-1)$$

Let's consider some examples.

Let's study the convergence of the following series

Example. $\frac{1}{3} + \frac{4}{3^2} + \frac{7}{3^3} + \frac{10}{3^4} + ...$

Solution. Let's find a general term of the series

$$u_n = \frac{a_n}{b_n} = \frac{1 + 3(n-1)}{3^n} = \frac{3n-2}{3^n}; \qquad u_{n+1} = \frac{3(n+1)-2}{3^{n+1}}.$$

Let's use the sign by d'Alembert

$$D = \lim_{n\to\infty} \frac{u_n + 1}{u_n} = \lim_{n\to\infty} \frac{(3n+1)3^n}{(3n-2)3 \cdot 3^n} =$$

$$=\frac{1}{3}\lim_{n\to\infty}\frac{3\frac{n}{n}-\frac{1}{n}}{3\frac{n}{n}-\frac{2}{n}}=\frac{1}{3}<1, \text{ the series converges}$$

Example. $\frac{2}{3}+\left(\frac{3}{6}\right)^2+\left(\frac{4}{9}\right)^3+\left(\frac{5}{12}\right)^4+...$

Solution. Let's find a general term of the series

$$u_n=\left(\frac{a_n}{b_n}\right)^n=\left(\frac{n+1}{3+3(n-1)}\right)^n=\left(\frac{n+1}{3n-3+3}\right)^n$$

Let's use the radical sign by Cauchy.

$$\sqrt{k}=\lim_{n\to\infty}\sqrt[n]{\left(\frac{n+1}{3n}\right)^n}=\lim_{n\to\infty}\frac{n+1}{3n}=\frac{1}{3}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)=$$

$$=\frac{1}{3}<1, \text{ the series converges.}$$

Example. $\frac{2}{5}+\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+...$

Solution. The series consists of terms of the infinitely decreasing geometric progression $a=\frac{2}{5}$; $q=\frac{1}{5}$ – is a denominator of the progression and its sum is

$S=\frac{a}{1-q}=\frac{\frac{2}{5}}{1-\frac{1}{5}}=\frac{2}{4}=\frac{1}{2}$, therefore the series converges.

The converges of the series $a+aq+aq^2+aq^3+...$ when $q<1$, presenting as an infinitely decreasing geometric progression we will present with the help of the sign by d'Alembert

$$u_n=aq^{n-1};\ u_{n+1}=aq^n;$$

$$D=\lim_{n\to\infty}\frac{aq^n}{aq^{n-1}}=\lim_{n\to\infty}\frac{q^{n-1}\cdot q}{q^{n-1}}=q<1, \text{ converges}$$

Example. $\frac{1}{2}+\frac{2}{7}+\frac{3}{12}+\frac{4}{17}+...$

Solution. Let's find a general term of the series

$$u_n = \frac{a_n}{b_n} = \frac{n}{2+5(n-1)} = \frac{n}{5n-3}.$$

Let's apply a sufficient sign divergence $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{n}{5n-3} = \frac{1}{5} \neq 0$.

The series diverges.

Example. $\sum\limits_{n=1}^{\infty} \frac{1}{3^n + 2}$.

Solution. $u_n = \frac{1}{3^n + 2}$. Let's apply the 1st sign of comparison. Let's compare with the series a general term of which is equal to $v_n = \frac{1}{3^n}$.

The series $\sum\limits_{n=1}^{\infty} \frac{1}{3^n}$ converges as an infinitely decreasing geometric progression and $u_n < v_n$, then the given series converges.

But if we consider the series $\sum\limits_{n=1}^{\infty} \frac{1}{3^n - 2}$ then for studying of convergence we must apply the 2nd sign of comparison.

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{3^n}{3^n - 2} = \left(\frac{\infty}{\infty}\right) = \lim_{n\to\infty} \frac{\frac{3^n}{3^n}}{\frac{3^n}{3^n} - \underset{\to 0}{\frac{2}{3^n}}} = 1 \begin{cases} < \infty \\ \neq 0 \end{cases}$$, both of the series behaves in the same way and the series $\sum\limits_{n=1}^{\infty} \frac{1}{3^n}$ converges. Therefore the series $\sum\limits_{n=1}^{\infty} \frac{1}{3^n - 2}$ converges as well.

Let's consider the generalized harmonic series $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$. For investigating of the convergence. Let's apply the integral sign by Cauchy

$$\int_1^{\infty}\frac{1}{x^p}dx=\lim_{b\to\infty}\int_1^{b}\frac{1}{x^p}dx=\begin{cases} p>1, =\lim\limits_{b\to\infty}\left.\dfrac{x^{-p+1}}{-p+1}\right|_1^b=\dfrac{1}{1-p}\lim\limits_{b\to\infty}\left(b^{1-p}-1^{1-p}\right)= \\ \qquad =\dfrac{1}{1-p}\lim\limits_{b\to\infty}\left(\dfrac{1}{b^{p-1}}-1\right)=\dfrac{1}{p-1} \\ p=1, =\lim\limits_{b\to\infty}\ln x\Big|_1^b=\lim\limits_{b\to\infty}\Big(\underbrace{\ln b}_{\searrow\infty}-\underset{0}{\underset{\parallel}{\ln 1}}\Big)=\infty,\ diverges \\ p<1, =\lim\limits_{b\to\infty}\left.\dfrac{x^{-p+1}}{-p+1}\right|_1^b=\dfrac{1}{1-p}\lim\limits_{b\to\infty}\Big(\underbrace{b^{1-p}}_{\searrow\infty}-1^{1-p}\Big)=\infty,\ diverges \end{cases}$$

Thus the series $\sum\limits_{n=1}^{\infty}\frac{1}{n^p}$ converges at $p>1$ and diverges at $p\le 1$

Example. $\sum\limits_{n=1}^{\infty}\frac{1}{n\sqrt{n+1}}$.

Solution. Let's compare it with the series $\sum\limits_{n=1}^{\infty}\frac{1}{n^{3/2}}$, that converges as the generalized harmonic series at $p=\frac{3}{2}>1$.

Let's apply the 2[nd] sign of comparison

$$\lim_{n\to\infty}\frac{n\sqrt{n}}{n\sqrt{n+1}}=\lim_{n\to\infty}\sqrt{\frac{\frac{n}{n}}{\frac{n}{n}+\underbrace{\frac{1}{n}}_{\searrow 0}}}=1,\begin{cases}\neq 0\\ <\infty,\end{cases}$$

Therefore both of the series behave in the same way i.d. converge.

Example. $\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\ldots$.

Solution. The given series obtained from the harmonic one diverges as a harmonic according to the theorem 1 by the dropping of the first eight terms.

Example. $\frac{100}{1!}+\frac{100^2}{2!}+\frac{100^3}{3!}+\ldots$

Solution. The general term of the series $u_n=\frac{100^n}{n!}$

Let's apply the sign by d'Alembert.

$$u_{n+1}=\frac{100^{n+1}}{(n+1)!};\qquad \begin{array}{l} n!=1\cdot 2:3\cdot\ldots\cdot n \\ (n+1)!=1\cdot 2:3\cdot\ldots\cdot n\cdot(n+1)\end{array}$$

$$D=\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{100\cdot \cancel{100^n}\cdot \cancel{n!}}{\cancel{100^n}\ \cancel{n!}\,(n+1)}=$$

$$=\left(\frac{100}{\infty}\right)=0,\ <1.\ \text{Converges}$$

Example. $\frac{1}{3ln3}+\frac{1}{4ln4}+\frac{1}{5ln5}+\ldots$

Solution. The general term of the series $u_n=\frac{2}{(n+2)ln(n+2)}$.

Let's apply the integral sign by Cauchy.

$$\int_1^{\infty}\frac{dx}{(x+2)ln(x+2)}=\left|\begin{array}{l} dx=d(x+2) \\ \frac{d(x+2)}{(x+2)}=d\,ln(x+2)\end{array}\right|=\lim_{b\to\infty}\int_1^b\frac{d\,ln(x+2)}{ln(x+2)}$$

$\lim_{b\to\infty} ln\,ln(x+2)\Big|_1^b=\left(\overbrace{ln\underbrace{ln\,b}_{\searrow\infty}}^{\nearrow\infty}-ln\,ln3\right)=\infty,$ diverges. Therefore the initial series diverges.

Example. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+1}$.

Solution. This is an alternating in sign series.

Let's apply the sign by Leibnitz.

1) $u_1>u_2>u_3>\ldots \qquad \frac{1}{2}>\frac{2}{2^2+1}>\frac{3}{3^2+1}>\ldots$

or $\frac{1}{2}>\frac{1}{2+\frac{1}{2}}>\frac{1}{3+\frac{1}{3}}>\frac{1}{4+\frac{1}{4}}>\ldots$

Such presentation of the general term allows to state that the term modules decrease.

2) $\lim\limits_{n\to\infty}\frac{n}{n^2+1}=\left(\frac{\infty}{\infty}\right)\lim\limits_{n\to\infty}\frac{\frac{n}{n}}{\frac{n\cdot n}{n}+\frac{1}{n}_{\searrow 0}}=\left(\frac{1}{\infty}\right)=0$

Therefore the series converges.

For investigation of the absolute as the conditional convergence let's consider,

the series composed of the term modules: $\sum\limits_{n=1}^{\infty}\frac{n}{n^2+1}$. Let's apply the 2nd sign of comparison. Let's compare with the series $\sum\limits_{n=1}^{\infty}\frac{1}{n}$, that diverges as a harmonic one

$u_n=\frac{n}{n^2+1}$; $v_n=\frac{1}{n}$.

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{n\cdot n}{n^2+1}=\left(\frac{\infty}{\infty}\right)=\lim_{n\to\infty}\frac{\frac{n^2}{n^2}}{\frac{n^2}{n^2}+\frac{1}{n}_{\searrow 0}}=1,\begin{cases}<\infty,\\ \neq 0.\end{cases}$$

i.d. both of the series behave in the same way − they diverge. It means that an alternating in sign series as a case of the alternating series converges conditionally.

Example. $1+\frac{1}{3}-\frac{1}{3^2}-\frac{1}{3^3}+\frac{1}{3^4}-\frac{1}{3^5}+...$

Solution. This is series with a random alternation in sign i.d. alternating series. For investigation of convergence, let's compose the series of modules

$$1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\frac{1}{3^4}+\frac{1}{3^5}+...$$

This series converges as an infinitely decreasing geometric progression.

Therefore the initial series converges absolutely.

Example. Find the sum of the series $\sum\limits_{n=1}^{\infty}\frac{1}{(n+3)(n+4)}$.

Solution. Let's present the general term of the series $u_n=\frac{1}{(n+3)(n+4)}$ as a sum of the simplest rational fractions with the help of indefinite coefficients $\frac{1}{(n+3)(n+4)}=\frac{A}{n+3}+\frac{B}{n+4}$.

Let's free from the denominator

$$1 = A(n+4) + B(n+3)$$

$$\begin{array}{l|l} n=-3 & 1 = A + 0;\ A = 1 \\ n=-4 & 1 = 0 - B;\ B = -1 \end{array}$$

i.d. $u_n = \frac{1}{(n+3)} - \frac{1}{(n+4)}$. Let's find a sum of the series

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\underbrace{\frac{1}{4} - \frac{1}{5}}_{n=1} + \underbrace{\frac{1}{5} - \frac{1}{6}}_{n=2} + \underbrace{\frac{1}{6} - \frac{1}{7}}_{n=3} + \ldots + \right.$$

$$\left. + \underbrace{\frac{1}{n+2} - \frac{1}{n+3}}_{n=n-1} + \underbrace{\frac{1}{n+3} - \frac{1}{n+4}}_{n}\right) = \lim_{n\to\infty}\left(\frac{1}{4} - \overset{\nearrow 0}{\frac{1}{n+4}}\right) = \frac{1}{4}.$$

As $\lim_{n\to\infty} S_n = \frac{1}{4}$ exist, the series converges.

13.2. Power series

Power series are the proper case of the functional series

$$u_1(x) + u_2(x) + u_3(x) + \ldots + u_n(x) + \ldots$$

Aggregation of x values for which there are functions and the series $\sum_{n=1}^{\infty} u_n(x)$ converges and it is called the domain of X convergence of the functional series. The sum of the functional series is as follows $S(x) = \lim_{n\to\infty} S_n(x)$

$$S(x) = S_n(x) + R_n(x)$$

A converging functional series is called an uniformly converging in a certain domain of x for any small assigned positive ε number there is such a positive N number that at $n \ge N$, $|R_n(x)| < \varepsilon$, $\forall x \in X$.

The sufficient sign of the uniform convergence of a functional series – is a sign by Weirstrass: $u_1(x)$, $u_2(x)$, $u_3(x)$,… $u_n(x)$,… of the absolute value do not exceed X positive terms a_1, a_2, a_3,…, a_n,… in a certain domain of X and besides the functional series $\sum_{n=1}^{\infty} a_n$ converges then the functional series $\sum_{n=1}^{\infty} u_n(x)$ in the domain converges uniformly.

We can integrate and differentiate uniformly converging series by terms.

Power series is a functional series presented as $a_0 + a_1(x-a) + a_2(x-a)^2 + ... + a_n(x-a)^n + ...,$ where a_0, $a_1,...,$ $a_n,...$ are coefficients of the power series.

Theorem by Abel. If a power series converges at $x = x_0$ then it converges and besides absolutely at any x value such as $|x-a| < |x_0 - a|$.

Consequence. Any power series has a convergence interval $|x-a| < R$ or $-R+a < x < R+a$ with a centre at the point inside which the series converges absolutely and outside it diverges. On the ends of the interval of convergence $x = a \pm R$ are different series that behave differently they can converge absolutely or conditionally or they diverge. Number R is a half of the convergence interval is called a convergence radius R can vary from zero to infinity.

For finding of a convergence radius we can use the following formulae

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|, \ R = \frac{1}{\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}} \text{ or}$$

we can apply the signs d'Alembert or Cauchy's while calculating the term series, i.d. to find a convergence radius from the given conditions

$$\lim_{n\to\infty}\frac{|u_{n+1}|}{|u_n|} < 1, \ \lim_{n\to\infty}\sqrt[n]{|u_n|} < 1, \text{ where}$$

$$a_0 = u_0, \ u_n = a_n(x-a)^n.$$

The series obtained by power series differentiation or integration by parts and they have the same convergence interval and their sum within the convergence interval is correspondingly equal to the derivative or the integral from the sum of the initial series.

Find the convergence domain of power series.

Example. $\sum_{n=1}^{\infty}\frac{1}{n^2}(x-1)^n$.

Solution. $a_n = \frac{1}{n^2}$; $a_{n+1} = \frac{1}{(n+1)^2}$;

$$R=\lim_{n\to\infty}\frac{|a_n|}{|a_{n+1}|}=\lim_{n\to\infty}\frac{(n+1)^2}{n^2}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2=1$$

$$|x-1|<1;\ -1<x-1<1,\ 0<x<2$$

Let's apply the sign by d'Alembert

$$u_n=\frac{(x-1)^n}{n^2};\ \ u_{n+1}=\frac{(x-1)^{n+1}}{(n+1)^2};$$

$$\lim_{n\to\infty}\frac{|u_{n+1}|}{|u_n|}=\lim_{n\to\infty}\frac{\left|(x-1)^n(x-1)\right|\left|n^2\right|}{\left|(n+1)^2\right|\left|(x-1)^n\right|}=$$

$$=|x-1|\cdot\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2=|x-1|<1;$$

$-1<x-1<1;\ 0<x<2$. The result is the same.

Let's investigate the series convergence on the end of the convergence interval $x=0$. We will obtain the series $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{n^2}$.

This is an alternating in sign series. It converges according to the sign by Leibnitz.

1) $u_1>u_2>u_3>...$ $\qquad \frac{1}{n^2}>\frac{1}{(n+1)^2};$

$(n+1)^2>n^2;\quad n^2+2n+1>n^2;\quad 2n+1>0$.

The first condition is executed.

2) $\lim\limits_{n\to\infty}u_n=\lim\limits_{n\to\infty}\frac{1}{n^2}=\frac{1}{\infty}=0$. The second condition is executed.

$x=2$. We will obtain a series $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}$. It converges as a generalized harmonic series at $p=2>1$.

Therefore the convergence domain is $X\in[0;2]$.

Example. $1!(x-3)+2!(x-3)^2+3!(x-3)^3+...$

Solution. Here $a_n=n!\quad a_{n+1}=(n+1)!$

$$R=\lim_{n\to\infty}\frac{|a_n|}{|a_{n+1}|}=\lim_{n\to\infty}\frac{n!}{(n+1)!}=\lim_{n\to\infty}\frac{1}{n+1}=\frac{1}{\infty}=0$$

The series converges only at $x-3=0$, i.d. at the point $x=3$.

Example. $\frac{x^2}{1!}+\frac{x^4}{2^2}+\frac{x^6}{3!}+...+\frac{x^{2n}}{n!}+...$

Solution. $a_n=\frac{1}{n!}$; $a_{n+1}=\frac{1}{(n+1)!}$

$$R=\lim_{n\to\infty}\frac{(n+1)!}{n!}=\lim_{n\to\infty}(n+1)=\infty.$$

Therefore the series converges at $x\in(-\infty,+\infty)$.

Consequence $\lim_{n\to\infty}\frac{x^{2n}}{n!}=0$, $\forall x\in(-\infty,+\infty)$.

Example. $\sum_{n=1}^{\infty}\frac{(x-2)^{n^2}}{n^n}$.

Solution. $u_n=\left[\frac{(x-2)^n}{n}\right]^n$. Let's apply the sign by Cauchy

$$\lim_{n\to\infty}\sqrt[n]{\left|\left[\frac{(x-2)^n}{n}\right]\right|}=\lim_{n\to\infty}\frac{|x-2|^n}{n}=\begin{cases}0, & при \ |x-2|\le 1\\ \infty, & при \ |x-2|>1\end{cases}$$

Therefore the series converges at $|x-2|\le 1$.

$-1\le x-2\le 1$, $1\le x\le 3$ i.d. $x\in[1;3]$.

Example. Find the sums of the series

a) $1+2x+3x^2+4x^3+...\quad |x|<1$

b) $x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+...\quad |x|<1$

Solution. Both these examples can be solved by differentiation and integration of the series $1+x+x^2+x^3+...\quad |x|<1$.

This series is an infinitely decreasing geometric progression with a sum $S=\frac{1}{1-x}$.

$$1+x+x^2+x^3+x^4...=\frac{1}{1-x}$$

Differentiating both parts of the given equality we will obtain

$$1+2x+3x^2+4x^3=\frac{1}{(1-x)^2},\quad x\in[-1;1)$$

Integrating in the limits from zero to x we are obtaining

$$x+x^2+x^3+x^4...=-ln|1-x|, \quad x\in[-1;1).$$

Any infinitely differentiating function on the interval $|x-x_0|<R$, i.d. $x\in(x_0-R,\, x_0+R)$ can be expanded into its converging infinite power Taylor series.

$$f(x)=f(x_0)+\frac{f'(x_0)(x-x_0)}{1!}+\frac{f''(x_0)}{2!}(x-x_0)^2+...+$$

$+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+...$ under the condition that remaining term

$R_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$ tends to zero, i.d.

$$\lim_{n\to\infty} R_n(x)=\lim_{n\to\infty}\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}=0, \text{ where}$$

$$c=x_0+\theta(x-x_0),\ 0<\theta<1.$$

at $x_0=0$ we are obtaining Maclaurin's series

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+...+\frac{f^{(n)}(0)}{n!}x^n+...$$

If at the certain interval, containing x_0, $|f^{(n)}(x)|<M$, then at any n, where M – is a positive constant then $\lim_{n\to\infty} R_n=0$ and $f(x)$ function is expanded into Taylor series. Some examples of expansion.

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+... \qquad x\in(-\infty,\infty)$$

$$sin\,x=\frac{x}{1!}-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+... \qquad x\in(-\infty,\infty)$$

$$cos\,x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+... \qquad x\in(-\infty,\infty)$$

$$ln(1+x)=x-\frac{x^2}{2!}+\frac{x^3}{3!}-\frac{x^4}{4!}+... \qquad x\in(-1,1]$$

$$arctgx=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+... \qquad x\in[-1,1]$$

Expand the following function into the power series.

Example. $f(x)=a^x$, $a>1$.

Solution. Let's substitute in the expansion $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+...$ x for $x\ln a$, we will obtain $e^{x\ln a}=a^x=1+x\ln a+\frac{x^2\ln^2 a}{2!}+\frac{x^3\ln^3 a}{3!}+...$ $\quad x\in(-\infty,\infty)$.

Example. $f(x)=\cos^2 x$.

Solution. Let's use the formula of degree decrease and cosine expansion $\cos^2 x=\frac{1+\cos 2x}{2}$

$$\cos^2 x=\frac{1}{2}+\frac{1}{2}\left(1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}-\frac{(2x)^6}{6!}\right)=$$

$$=1-\frac{2x^2}{2!}+\frac{2^3x^4}{4!}-\frac{2^5x^6}{6!}+...,\ x\in(-\infty,\infty).$$

Example. $f(x)=\sin^2 x$.

Solution. Let's find a function value and its derivatives at the point $x=0$

$$f(x)=\sin^2 x, \qquad f(0)=0$$

$$f'(x)=2\sin x\cos x=\sin 2x, \quad f'(0)=0$$

$$f''(x)=2\cos 2x=2\sin\left(2x+\frac{\pi}{2}\right), \qquad f''(0)=2$$

$$f'''(x)=-2^2\sin 2x=2^2\sin\left(2x+2\frac{\pi}{2}\right), \qquad f'''(0)=0$$

$$f^{IV}(x)=-2^3\cos 2x=2^3\sin\left(2x+3\frac{\pi}{2}\right), \qquad f^{IV}(0)=-2^3$$

$$f^n(x)=2^{n-1}\sin\left[2x+\frac{\pi}{2}(n-1)\right]$$

$$f^{(n+1)}(x)=2^n\sin\left[2x+n\frac{\pi}{2}\right], \quad f^{n+1}(c)=2^n\sin\left[2c+\frac{n\pi}{2}\right]$$

$$R_n=\frac{2^n\sin\left(2c+\frac{n\pi}{2}\right)}{(n+1)!}\cdot x^{n+1}=\frac{2^n x^n\cdot x\cdot 2\sin\left(2c+\frac{n\pi}{2}\right)}{2(n+1)!}\le\frac{1}{2}\frac{(2x)^{n+1}\cdot 1}{(n+1)!},$$

as $\sin\left(2c+\frac{n\pi}{2}\right)\le 1$.

$\lim\limits_{n\to\infty} R_n = \frac{1}{2}\lim\limits_{n\to\infty}\frac{(2x)^{n+1}}{(n+1)!} = 0$ at any x, and a function $f(x) = \sin^2 x$ can be expanded into degree series

$$\sin^2 x = \frac{2}{2!}x^2 - \frac{2^3}{4!}x^4 + \frac{2^5}{6!}x^6 - \frac{2^7}{8!}x^8 + ...$$

Such expansion can be obtained as in the previous example.

13.3. Fourier's Series

The function $f(x)$ satisfying on the interval $[-\pi;\pi]$ the conditions by Dirichlet i.d. continuous or having a final number of break points of the first order and having the final number of extremums can be expanded into converging to it Fourier's series on this interval.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_m \cos mx + b_m \sin mx)\text{, where}$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx dx,\ (m = 0,1,2,...)$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx dx,\ (m = 1,2,3,...)$$

The sum of Fourier's series is a periodic function with a period 2π.

In the break points of the first order x_0 a function is $f(x_0) = \frac{1}{2}[f(x_0 - 0) + f(x_0 + 0)]$.

Let's consider $f(\pm\pi) = \frac{1}{2}[f(-\pi+0) + f(\pi-0)]$ for the function value on the segment border $[-\pi;\pi]$.

If the function $f(x)$ is assigned on the segment $[-l;l]$, where l − is a random number then if the condition by Dirichlet are executed $f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\frac{m\pi x}{l} + b_m \sin\frac{m\pi x}{l}\right)$, where $a_m = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{m\pi x}{l}dx$, $b_m = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{m\pi x}{l}dx$.

If a function $f(x)$ is an even one then $f(x)=\frac{a_0}{2}+\sum_{m=1}^{\infty} a_m \cos\frac{m\pi x}{l}$, where

$a_m=\frac{2}{l}\int_0^l f(x)\cos\frac{m\pi x}{l}dx$, $b_m=0$.

If a function $f(x)$ is an odd one, then $f(x)=\sum_{m=1}^{\infty} b_m \sin\frac{m\pi x}{l}$, where

$b_m=\frac{2}{l}\int_0^l f(x)\sin\frac{m\pi x}{l}dx$, $a_m=0$.

If function $f(x)$ is assigned on the interval $[ol]$, then it must be redefine for $[-lo]$ in order to be expanded into a Fourier's series.

Besides if we have $f(x)$ at random, then in expansion there will be both coefficients of a_m and b_m.

If we redefine it evenly then there will be only a_m and $b_m=0$.

If we redefine oddly then there will be only b_m and $a_m=0$.

Example. Expand the function $f(x)=-x+\frac{\pi}{2}$ assigned on the interval $[-\pi;\pi]$ into Fourier's series. Draw a diagram of the sum of a Fourier's series.

Solution. Function of the general view

$$f(x)=\frac{a_0}{2}+\sum_{m=1}^{\infty}(a_m \cos mx+b_m \sin mx)$$

Diagram of the function

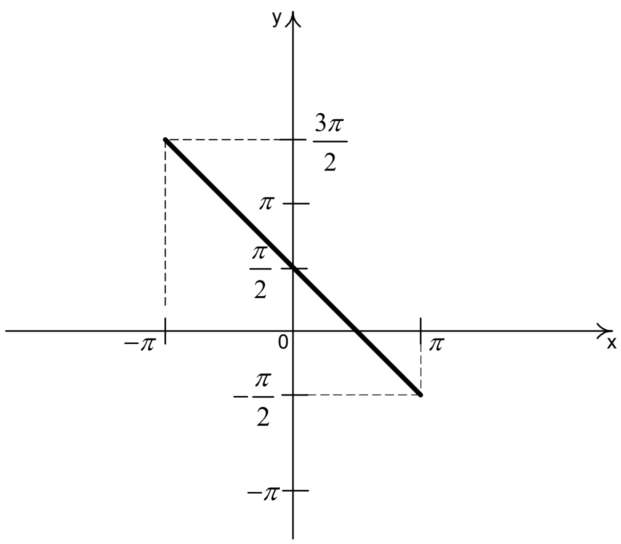


Fig.48

Diagram of the sum of Fourier's series

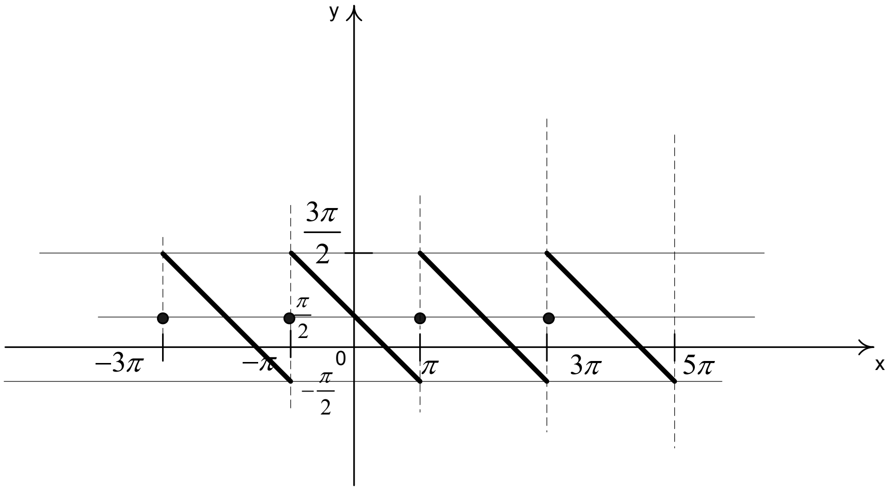


Fig.49

Let's find the coefficients $a_0=\frac{1}{\pi}\int\limits_{-\pi}^{\pi}\left(-x+\frac{\pi}{2}\right)dx=$

$$=\frac{1}{\pi}\left[-\frac{x^2}{2}\Bigg|_{-\pi}^{\pi}+\frac{\pi}{2}x\Bigg|_{-\pi}^{\pi}\right]=\frac{1}{\pi}\left[-\frac{1}{2}\left(\pi^2-(\pi)^2\right)+\frac{\pi}{2}(\pi+\pi)\right]=\pi$$

$$a_m=\frac{1}{\pi}\int\limits_{-\pi}^{\pi}\left(-x+\frac{\pi}{2}\right)\cos mxdx=-\frac{1}{\pi}\int\limits_{-\pi}^{\pi}x\cos mxdx+\frac{1}{2}\int\limits_{-\pi}^{\pi}\cos mxdx=$$

$$=\begin{vmatrix}x=u\\ \cos mxdx=dv\\ du=dx\\ v=\dfrac{1}{m}\sin mx\end{vmatrix}=-\frac{1}{\pi}\left[x\frac{1}{m}\underset{\underset{0}{\parallel}}{\sin mx}\Bigg|_{-\pi}^{\pi}-\frac{1}{m}\int\limits_{-\pi}^{\pi}\sin mxdx\right]+$$

$$+\frac{1}{2m}\underset{\underset{0}{\parallel}}{\sin}mx\Bigg|_{-\pi}^{\pi}=\frac{1}{m\pi}\cos mx\Bigg|_{-\pi}^{\pi}=0;$$

$$b_m=\frac{1}{\pi}\int\limits_{-\pi}^{\pi}\left(-x+\frac{\pi}{2}\right)\sin mxdx=\frac{-1}{\pi}\int\limits_{-\pi}^{\pi}x\sin mxdx+\frac{1}{2}\overset{\overset{0}{\parallel}}{\int\limits_{-\pi}^{\pi}\sin mxdx}=$$

$$=\begin{vmatrix}x=u\\ \sin mxdx=dv\\ dx=du\\ v=\dfrac{-1}{m}\cos mx\end{vmatrix}=-\frac{1}{\pi}\left[\frac{-x}{m}\cos mx\Bigg|_{-\pi}^{\pi}+\frac{1}{m}\int\limits_{-\pi}^{\pi}\underset{\underset{0}{\parallel}}{\cos mxdx}\right]=$$

$$= \frac{1}{m\pi}\left[\pi \cos m\pi - (\pi)\cos m(\pi)\right] = \frac{2\cos m\pi}{m} = \frac{2(-1)^n}{m};$$

While finding the coefficients we use the integration by parts taking into account that an integral from the odd function in symmetric limits is equal to zero, $\sin m\pi = 0$ and $\cos m\pi = (-1)^m$.

$$f(x) = \frac{\pi}{2} + \sum_{m=1}^{\infty}\left(0\cos nx + \frac{z(-1)^m}{m}\sin mx\right) =$$

$$= \frac{\pi}{2} + 2\left(-\frac{1}{-1}\sin x + \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x\ldots\right).$$

$$-x + \frac{\pi}{2} = \frac{\pi}{2} - 2\left(\frac{simx}{1} - \frac{sim2x}{2} + \frac{sim3x}{3} - \frac{sim4x}{4} + \ldots\right).$$

Then $x = 2\left(\frac{simx}{1} - \frac{sim2x}{2} + \frac{sim3x}{3} - \frac{sim4x}{4} + \ldots\right)$

Example. Expand the function $f(x) = -\pi$ into Fourier's series assigned on the interval $[0,\pi]$.

Solution. Diagram of the function

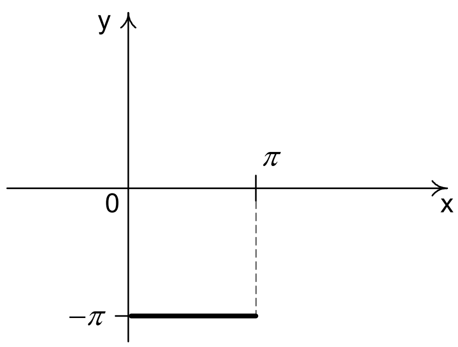


Fig.50

Let's redefine function on the interval $[-\pi;0]$ oddly and present a diagram of the sum of a Fourier's series.

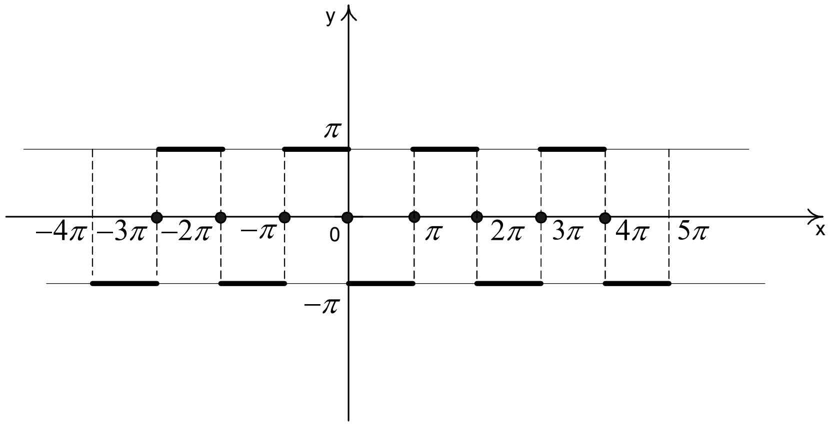


Fig.51

$$f(x)=\sum_{m=1}^{\infty} b_m \sin mx, \qquad a_m = 0 \quad (m=0,1,2...)$$

$$b_m = \frac{2}{\pi}\int_0^{\pi}(-\pi)\sin mxdx = \not{\pi}2\frac{1}{m}\cos mx\Big|_0^{\pi} = \frac{2}{m}\left((-1)^m - 1\right) = \begin{cases} -\dfrac{4}{m}, & m - \text{is odd} \\ 0, & m - \text{is even} \end{cases}$$

$$f(x) = -4\sum_{m=1}^{\infty}\frac{1}{m}\sin mx = -4\left(\frac{\sin x}{1}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+...\right).$$

Example. Expand the function assigned with the help of the diagram into Fourier's series

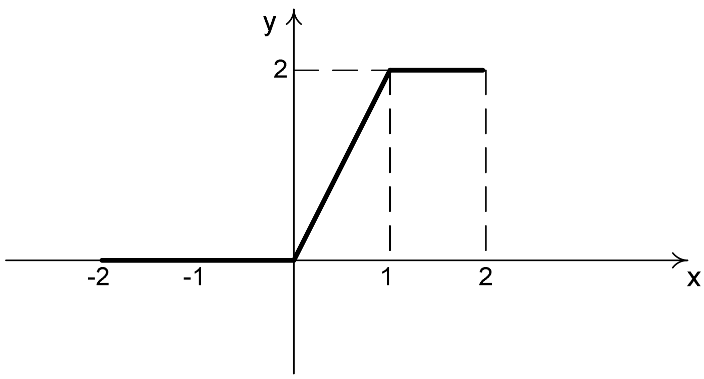


Fig.52

Let's write down this function analytically

$$f(x)=\begin{cases} 0, & x\in[-2;0] \\ 2x, & x\in[0;1] \\ 2, & x\in[1;2] \end{cases}$$

Function of the general view

$l = 2$

$$f(x)=\frac{a_0}{2}+\sum_{m=1}^{\infty}\left(a_m\,cos\frac{m\pi x}{2}+b_m\,sin\frac{m\pi x}{2}\right)$$

$$a_0=\frac{1}{2}\left[\int_{-2}^{0}0dx+\int_{0}^{1}2xdx+\int_{1}^{2}2dx\right]=\frac{1}{2}\left[2\frac{x^2}{2}\bigg|_0^1+2x\bigg|_1^2\right]=\frac{3}{4}$$

$$a_m=\frac{1}{2}\left[\int_{-2}^{0}0cos\frac{m\pi x}{2}dx+2\int_{0}^{1}=cos\frac{m\pi x}{2}dx+2\int_{1}^{2}cos\frac{m\pi x}{2}dx\right]=$$

$$=\left|\begin{array}{l}x=u;\ du=dx\\ cos\frac{m\pi x}{2}dx=dV\\ V=\frac{2}{m\pi}sin\frac{m\pi x}{2}\end{array}\right|=\frac{1}{2}\left[2x\cdot\frac{2}{m\pi}sin\frac{m\pi x}{2}\bigg|_0^1-2\cdot\frac{2}{m\pi}\int_0^1 sin\frac{m\pi x}{2}dx+\right.$$

$$\left.+\frac{4}{m\pi}sin\frac{m\pi x}{2}\bigg|_1^2\right]=\frac{1}{2}\left[\frac{4}{m\pi}\cancel{sin\frac{m\pi}{2}}+\frac{4}{m^2\pi^2}cos\frac{m\pi x}{2}\bigg|_0^1-\frac{4}{m\pi}\cancel{sin\frac{m\pi}{2}}\right]=$$

$$=\frac{2}{m^2\pi^2}\left(cos\frac{m\pi}{2}-1\right)=\frac{-2}{m^2\pi^2};$$

$$b_m=\frac{1}{2}\left[\int_{-2}^{0}0sin\frac{m\pi x}{2}dx+2\int_{0}^{1}x\,sin\frac{m\pi x}{2}dx+2\int_{1}^{2}sin\frac{m\pi x}{2}dx\right]=$$

$$=\left|\begin{array}{l}x=u;\ dx=du\\ sin\frac{m\pi x}{2}dx=dV\\ V=\frac{2}{m\pi}cos\frac{m\pi x}{2}\end{array}\right|=\frac{1}{2}\left[-2x\cdot\frac{2}{m\pi}cos\frac{m\pi x}{2}\bigg|_0^1\not+2\cdot\frac{2}{m\pi}\int_0^1 sin\frac{m\pi x}{2}dx-\right.$$

$$\left.-2\frac{2}{m\pi}cos\frac{m\pi x}{2}\bigg|_1^2\right]=\frac{1}{2}\left[-\frac{4}{m\pi}\underbrace{\xcancel{cos\frac{m\pi}{2}}}_{0}+\frac{4\cdot 2}{m^2\pi^2}sin\frac{m\pi x}{2}\bigg|_0^1-\right.$$

$$\left.-\frac{4}{m\pi}\left(cos\,m\pi-\underbrace{\xcancel{cos\frac{m\pi}{2}}}_{0}\right)\right]=\frac{4}{m^2\pi^2}sin\frac{m\pi}{2}-\frac{2}{m\pi}(-1)^m$$

Let's consider the value $sin\frac{m\pi}{2}$ for different values of m.

$$\sin\frac{m\pi}{2}=\begin{cases}1, & m=1,\ 5,\ 9,\ ..\\ 0, & m=2,\ 6,\ 10,\ ...\\ -1, & m=3,\ 7,\ 11,\ ...\end{cases}$$

Thus $f(x)=\frac{3}{8}-\frac{2}{\pi^2}\left(\frac{\cos\frac{\pi x}{2}}{1^2}+\frac{\cos\frac{2\pi x}{2}}{2^2}+\frac{\cos\frac{3\pi x}{2}}{3^2}+...\right)+$

$$+\frac{4}{\pi^2}\left(\frac{\sin\frac{\pi x}{2}}{1^2}-\frac{\sin\frac{3\pi x}{2}}{3^2}+\frac{\sin\frac{5\pi x}{2}}{5^2}-\frac{\sin\frac{7\pi x}{2}}{7^2}+...\right)+\frac{2}{\pi}\left(\frac{\sin\frac{\pi x}{2}}{1}-\right.$$

$$\left.-\frac{\sin\frac{2\pi x}{2}}{2}+\frac{\sin\frac{3\pi x}{2}}{3}-\frac{\sin\frac{4\pi x}{2}}{4}+...\right).$$

Home task

Investigate the convergence of the numeric series with positive terms

1. $\sum_{n=1}^{\infty}\frac{3n-1}{\sqrt{n2^n}}$ 2. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^3-2n+2}}$ 3. $\sum_{n=1}^{\infty}\left(\frac{n}{n+7}\right)^{n^2}$

4. $\sum_{n=1}^{\infty}\frac{1}{(3n-1)ln(3n-1)}$ 5. $\sum_{n=1}^{\infty}\left(arcsin\frac{1}{3n}\right)^{2n}$ 6. $\sum_{n=1}^{\infty}\frac{(n+2)!}{n^n}$

Find the sum of the series

1. $\sum_{n=1}^{\infty}\frac{1}{(n+4)(n+5)}$ 2. $\sum_{n=1}^{\infty}\frac{5^n-3^n}{15^n}$ 3. $\sum_{n=1}^{\infty}\frac{1}{(3n-1)(3n+2)}$

Investigate the alternating in sign for convergence and absolute convergence.

1. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{3^n}$ 2. $\sum_{n=1}^{\infty}(-1)^n\frac{n+1}{n^2}$ 3. $\sum_{n=1}^{\infty}(-1)^n \sin^n\frac{\pi}{5n}$

4. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3}{\sqrt{n}\sqrt{(n+3)^3}}$ 5. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{(n+4)ln^2(n+4)}$

Find the domain of the power series

1. $\sum_{n=1}^{\infty} \frac{x^n}{n2^n}$ 2. $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$ 3. $\sum_{n=1}^{\infty} \frac{1}{n(x-2)^n}$

4. $\sum_{n=1}^{\infty} \frac{(x-3)^n}{2^n}$ 5. $\sum_{n=1}^{\infty} \frac{(x-4)^n}{n^2}$

Expand into Maclaurin series. Define the domain of the series

1. $f(x) = cos\,3x$; 2. $f(x) = \frac{x^2}{1+x^2}$; 3. $f(x) = \frac{2}{x+1}$

Expand the function assigned on the interval $[-\pi;\pi]$ into Fourier's series

1. $f(x) = \begin{cases} 0, & x \in [-\pi;0] \\ x+1, & x \in [0;\pi] \end{cases}$ 2. $f(x) = \frac{|x|}{2}$

3. $f(x) = \begin{cases} 1-\frac{x}{2}, & x \in [-\pi;0] \\ 0, & x \in [0;\pi] \end{cases}$ 4. $f(x) = x^2 - \pi^2$

THEMATIC UKRAINIAN - ENGLISH DICTIONARY

1. Аналітична геометрія на площині

біля, по відношенню до	[əntu]	onto
більше ніж	[´i:kwələ(r) ðen]	equalor then
вершина	[´eipeks]	apex
висота	[ǽltitju:d]	altitude
виходячий, проведений (з)	[dropt]	dropped
відношення	[´reiʃiəu]	ratio
вісь (вісі) (геом.)	[ǽksis (æksi:z)]	axis (axes)
вісь (фіз.)	[ǽksl]	axle
внутрішній кут	[´inə ǽngl]	inner angle
загальний	[´dʒenərəl]	general
значення	[´vælju:]	value
крива	[kə:v]	curve
кутовий коефіцієнт	[ǽngjulə ´fæktə]	angular factor
площа	[εəria]	area
проекція	[prə´dʒekʃən]	projection
рівняння	[ik´weiʃən]	equation
сітка	[grid]	grid
шкала, масштаб	[skeil]	scale

2. Елементи математичного аналізу

приймати	[ək´sept]	accept
виділити	[´pikiŋ aut]	picking out
висновок	[kən´klu:ʒn]	conclusion
граничний перехід	[´pæsidʒ tu ðə limit]	passage to the limit

додати та відняти	[æd ənd di´dʌkt]	add and deduct
дріб	[´frækʃn]	fraction
заміна, підстановка	[,sʌbsti´tju:ʃən]	substitution
знак	[sain]	sign
зручний	[kən´vinjə:nt]	convenient
квадратний тричлен	[sk´wεə trai´nəumiəl]	square trinomial
невизначеність	[ikwivə´keiʃn]	equivocations
невизначеність ($\frac{0}{0}$, $\frac{\infty}{\infty}$)	[ʌn´sə:tenti]	uncertainty
незмінний	[ʌn´tʃeindʒd]	unchanged
неперервний	[kən´tinjuəs]	continuous
нерівність	[,ini: ´kwoliti]	inequality
нескінченно велика (н.в.)	[in´definitli la:dʒ]	indefinitely large
нескінченно мала (н.м.)	[infinitəsiməl]	infinitesimal
обережно	[,pə´tikjulə´kεəful]	particular careful
обернений	[in´və:s]	inverse
обертаючийся	[tə:niŋ]	turning
порівняння	[kəm´pærisn]	comparison
послідовно	[´si:kwənʃiəli]	sequentially
прямує	[´əimiŋ tu]	aiming to
роблячи	[du:iŋ]	doing
розкриття	[di´pləi:ment]	deployment
саме	[´neimli]	namely
слідство, висновок	[´kənsikwəns]	consequence
спряжений, сполучений	[´kəndʒugeit]	conjugate
тотожні перетворення	[aidentikəl kən´və: ʃ(ə)n]	identical conversions
x прямує до 0 ($x \to 0$)	[iks tənd tə zirəu]	x tend to 0

чудовий (мат.)	[riˊma:kəbl]	remarkable

3. Основи диференціального числення функції однієї незалежної змінної

вимагати	[riˊkwaiə]	require
вимога	[riˊkwaiəmənt]	requirement
висновок	[kənˊklu: ʒ(e)n]	conclusion
висновок	[risˊtrikt]	restrict
відносно	[ˊrelətivli]	relatively
достатній	[saˊfiʃnt]	sufficient
досягає	[əˊtʃi:vs]	achieves
дотична	[ˊtændʒ(ə)nt]	tangent(line)
заміна	[ˊsʌbstitju:t]	substitute
застосовується	[əˊplaid]	applied
звідки	[wens]	whence
зростає	[inˊkri:siz]	increases
кінець (обмеження)	[ˊbɔ:də]	border
коло	[ˊneibəhud]	neighborhood
момент часу	[ˊinstənt]	instant
моментальний	[ˌinst(ə)nˊteinjəs]	instantaneous
монотонність	[mənəˊtənəsiti]	monotonicity
не дорівнююча нулю границя	[nɔnˊtriviəl limit]	nontrivial limit
необмежено	[ˌʌnristˊrictidli]	unrestrictedly
непарний	[ˊɔdnes]	oddness
неперіодична	[əˊsiklik]	aciclic

нескінченність	[ˌpə:piˊtju:iti]	perpetuity
неявний	[imˊplisit]	implicit
нижче	[biˊləu]	below
нормаль	[ˊnɔ:m(ə)l]	normal(line)
однозначний	[singl vælju:d]	single-valued
одразу, водночас	[iˊmi:djətli]	immediately
опуклість	[ˊkɔnˊveksiti]	convexity
парний	[ˊpæriti]	parity
перегиб	[inˊflekʃn]	inflection
похідна	[diˊrivətiv]	derivative
початковий	[iˊniʃ(ə)l]	initial
приріст	[ˊinkrimənt]	increment
прямувати до	[əˊprəutʃiz tu]	approaches to
прямувати до нуля	[eimz tu ˊzir əu]	aims to zero
прямуючий	[si:kiŋ]	seeking
розкриття	[disˊkləuziŋ]	disclosing
серед	[əˊmʌŋ]	among
спадає	[dikˊri:s(iz)]	decreases
створити	[kri:ˊeit]	to create
тотожність	[aiˊdentiti]	identity
угнутість	[ˊkɔnˊkæviti]	concavity
ще раз	[wʌns əˊgen]	once again

4. Невизначений інтеграл. Методи інтегрування

безпосередній, прямий	[daiˊrəkt]	direct
велика літера $F(x)$	[kæpitl]	capital (upper) $F(x)$
виділити	[iksˊtrækt]	extract

властивості	[´prɔpətis]	properties
дійсний корінь	[riəl ru:t]	real root
добуток	[´prɔdʌkt]	product
доданок	[sʌmænd]	summand
залишок	[ri´meində]	remainder
заміна змінної	[´tʃeindʒ ɔv ´veriəbl]	change of variable
звільнення	[kʌn´və: ʃn]	conversion
зворотній інтеграл	[ritə:n´intigr(ə)l]	return integral
знаменник	[dinɔminetə(r)]	denominator
інваріантність, незмінність	[inveri´eiʃən]	invariation
комплексно-спряжені корені	[´kʌmpleks - ´kɔndʒugeitid (ru:ts)]	complex – conjugated (roots)
косинус	[kəu´sain]	cosine
котангенс	[´kəutændʒent]	cotangent
кратний	[´mʌltipl]	multiple
лінійні алгебраїчні рівняння	[´lainə ældʒi´breik i´kweiʃnz]	linear algebraic equations
мала літера $f(x)$	[lou]	low $f(x)$
множник	[mʌlti´plaiə(r)]	multiplier
наданий	[di´zaineitid]	designated
невизначений	[in´definit]	indefinite
невизначений коефіцієнт	[ʌndi´tə:mind ,kəui´fiʃənt]	undetermined coefficient
непарне число	[ɔd´nʌmba]	odd number
неповторний	[nɔn-ri´kə:riŋ]	non-recurring
неправильний дріб	[imprɔpə´frækʃn]	improper fraction
парне число	[i:vən´nʌmba]	even number

первісна	[əntideˊrivətiv]	antiderivative
підінтегральна функція	[ˊintəgrand]	integrand
підкореневий вираз	[sʌbˊrædikl iksˊpreʃn]	subradical expression
підстановка	[sʌbstiˊtju:ʃn]	substitution
повний квадрат	[kəmˊplit skwɛə]	complete square
подібний	[ˊsimilə]	similar
показник	[iksˊponənt]	exponent
поліном, многочлен	[polinɔmiəl]	polynomial
помножити	[ˊmʌltiplai]	multiply
правильний дріб	[ˊprɔpəˊfrækʃn]	proper fraction
раціональний дріб	[ˊræʃən(ə)l frækʃn(z)]	rational fractions
розклад	[disˊklɔuʒə]	disclosure
розкласти	[iksˊpænd]	expand
синус	[sain]	sine
спрощений	[simpliˊfaid]	simplified
ступінь	[diˊgri:]	degree
тангенс	[ˊtændʒənt]	tangent
тригонометричні підстановки	[trigənəmetrik sʌbstiˊtju: ʃn]	trigonometric substitution
у ступені	[inði:ˊpauə]	in the power
універсальний	[ˊdʒenərəl]	general
ціла частина	[intəgrəl pa:t]	integral part
ціле	[ˊintidʒə]	integer
чисельник	[ˊnɔmineitə(r)]	nominator

5. Визначений інтеграл

арка циклоїда	[a:(r)k ɔv´saiklɔid]	arc of cycloid
гладка крива	[smu:θ kə:(r)v]	smooth curve
довжина дуги	[a:(r)k lenθ]	arc length
криволінійна	[ˌkə:vi´liniə]	curvilinear
момент інерції	[inə:rʃia ´məument]	inertia moment
невласний	[im´prɔpə]	improper
необмежена функція	[ʌn´baundid ´fʌŋkʃən]	unbounded function
непарна функція	[ɔd ´fʌŋkʃən]	odd function
нескінченний розрив	[in´finit breik]	infinite break
оцінка	[esti´meiʃən]	estimation
парна функція	[´i:vən ´fʌŋkʃən]	even function
поперечний переріз	[krɔs´ səkʃ'n)]	cross-section
статичний момент	[´stætik(ə)l ´məument]	statical moment
тіло обернення	[rəu´teiʃn ´sɔlid]	rotation solid
трапеція	[træpizɔid]	trapezoid
чверть	[´kwə:tə]	quarter
що підтверджується	[ðət iz kənfə:md bai]	that is confirmed by

6. Елементи лінійної алгебри

алгебраїчне доповнення	[ældʒi´breik kə´fæktə]	algebraic cofactor
лінійне перетворення	[´liniə trænsfɔ:meiʃn]	linear transformation
мінор	[mainə]	minor
нульова матриця	[´zirou ´meitriks]	zero matrix
одинична матриця	[ju:nit ´meitriks]	unit matrix

розкласти	[iks´pænd]	expand
стовбець	[´kɔləm]	column
строка	[rou]	row
у відповідності до	[ə´kɔ:(r)din tu]	according to

7. Елементи векторної алгебри

зв'язаний	[ri´leitid]	related
компланарний	[´cʌmplenə cʌm´pliniə]	complanar, complinear
многокутник	[pɔli´gɔnik]	polygonic
модуль	[´mɔdju:l]	module
напрямок	[di´rekʃn]	direction
орти	[ɔ(r)ts]	orts
позначається як, позначений	[´ma:(r)kt əz]	marked as
проекція	[prə´dʒekʃ(ə)n]	projection
протилежний	[´ɔpəzit]	opposite
розподільча властивість	[ðə´prɔpeti ɔv distri´bju: ʃ n]	the property of distribution
скалярний, скаляр	[´skæla:]	scalar
сполучений	[ædiʃn(ə)l]	additional
точка простору	[´speiʃl ´pɔint]	spatial point
що перенесений	[træns´fə:d]	transferred

8. Аналітична геометрія у просторі

вільний член	[´æbsəlu:t tə:m]	absolute term
гіперболічний параболоїд	[haipə:bəlik pe´ræbəlɔid]	hyperbolic paraboloid
гіперболічний циліндр	[haipə:bəlik ´silində]	hyperbolic cylinder
двопалий гіперболоїд	[tu ´ʃi:tid hai´pə:bəlɔid]	two-sheeted hyperboloid
еліпсоїд	[ilipsɔid]	ellipsoid
еліптичний параболоїд	[i´liptik pe´ræbəlɔid]	elliptic paraboloid
еліптичний циліндр	[i´liptik ´silində]	elliptic cylinder
жмуток площин	[pensil ov pleins]	pencil of planes
зв’язка площин	[bʌndl ov pleins]	bundle of planes
конгруентні відрізки	[´koun]	cone
конус	[´kɔŋgruəs intə:´septs]	congruous intercepts
нормальне рівняння	[´nɔ:məl ikweiʃn]	normal equation
нормувальний множник	[´nɔ:məlaiz mʌltiplaiə(r)]	normalize multiplier
однопалий гіперболоїд	[´si ŋgl-ʃæt hai´pə:bəlɔid]	single-shat hyperboloid
основа перпендикуляра	[´beisis of pə:pen´dikjulə]	basis of the perpendicular
параболічний циліндр	[peræbolik ´silində]	parabolic cylinder
параболоїд	[pe´ræbəlɔid]	paraboloid
поверхня обертання	[s´ə:fis ov revə´lu: ʃn]	surface of revolution
сфера	[sfiə]	sphere
твірна	[´fɔ:miŋ]	forming
уявна поверхня	[i´mædʒin(ə)ri ´sə:fis]	imaginary surface

циліндр	[silindrikəl ˊsə:fis]	cylindrical surface
циліндрична поверхня	[ˊsilində]	cylinder

9. Функції декількох незалежних змінних

відображення	[mæpiŋ]	mapping
концентричні сфери	[kʌnˊsentrik ˊsfiəz]	concentric spheres
лінії рівня	[ˊlevel ˊlain(z)]	level lines
найбистріший	[ðə ˊfastəst]	the fastest
найбільший	[ðə greitəst]	the greatest
найменший	[ðə li:st]	the least
непуста множність	[nʌnˊempti set]	nonempty set
неявна функція	[imˊplisit ˊfʌŋkʃn]	implicit function
поверхні рівня	[ˊlevel ˊsə(r)fisiz]	level surfaces
складова функція	[kɔmpleks ˊfʌnk ʃn]	complex function
стаціонарний	[ˊsteiʃn(ə)ri]	stationary
часткові похідні	[ˊpa:ʃ(ə)l deˊri:vətivz]	partial derivatives

10. Звичайні диференціальні рівняння

вимір	[,ˊmeʒəment]	measurement
відокремлювані змінні	[diˊvaidiŋ ˊvɛəriəblz]	dividing variables
комплексно-спряжені корені	[kɔmpleks kɔndʒugeit ru:ts]	complex-conjugate roots
кратний корінь	[ˊmʌltipl ru:t]	multiple root
метод варіації довільної сталої	[vɛəriˊeiʃənl meθɔd ɔv ðə ˊrændəm ˊkɔnstənt]	variational method of the random constant
метод підбору	[meθɔd ɔv insˊpekʃn]	method of inspection

неоднорідне рівняння	[inhɔmə´dʒi:nəs ikweiʃn]	inhomogenous equation
однорідне рівняння	[hɔmə´dʒi:nəs ikweiʃn]	homogenous equation
початкові умови	[iniʃiəl kən´diʃn]	initial condition
характеристичне рівняння	[´kærektəristik ikweiʃn]	characteristic equation
часткове рішення	[pa:(r)´tikju:lə sɔ´lu:ʃn]	particular solution

11. Подвійні інтеграли

взаємно-однозначна відповідність	[siŋgl-´vælid ,kɔris´pɔndəns]	single-valied correspondence
геометричний зміст	[dʒiɔ´metrik ´mi:niŋ]	geometric meaning
двократний інтеграл	[itəreitid integrəl]	iterated integral
замкнена область	[´klouzd dɔ´mein]	closed domain
криволінійні координати	[kə:v-laind kəu´ɔ:dinit(s)]	curve-lined coordinates
момент інерції	[i´nə:rʃiə moumənt]	inertia moment
плоска пластівка	[plein pleit]	plane plate
повторний інтеграл	[ripi:tid ´integrəl]	repeated integral
подвійний інтеграл	[´dʌbl ´integrəl]	double integral
полярні координати	[´pɔla:(r) kəu´ɔ:dinit(s)]	polar coordinates
статичний момент	[stə´tik moumənt]	static moment
центр ваги	[´græviti ´sent(r)ə]	gravity centre
циліндрична поверхня	[silindrikəl ´sə:fis]	cylindrical surface

12. Криволінійні інтеграли

астроїда	[´æstrɔid]	astroid
гладка крива	[smu:θ kə:v]	smooth curve
довжина дуги	[a(r)k lenθ]	arc length
елементарна дуга	[eli´mentəri a´(r)k]	elementary arc
змінна густина	[vɛəriəbl densiti]	variable density
контур інтегрування	[intəg´reiʃn ´kɔntuə]	integration contour
криволінійний інтеграл і роду	[´kɔntuə ´integrəlz ɔv ðə ´fə:st ´kaind]	contour integrals of the I[st] kind
криволінійний інтеграл ії роду	[´kɔntuə ´integrəlz ɔv ðə ´sekənd ´kaind]	contour integrals of the II[st] kind
криволінійний путь	[kə:vd wei]	curved way
кусково-гладка просторова крива	[,pi:s´waiz ´smu:θ ´speiʃl kə:v]	piecewise smooth spatial curve
ломана лінія	[´broukn lain]	broken line, break line
однозв’язна область	[simpli kə´nəktid dɔ´mein]	simply connected domain
однозначна функція	[van vælju:d ´fʌnkʃn]	one valued function
при пробігу контура K	[ət də´tuə ə´lɔŋ ðə ´kɔntuə K]	at detour along the contour K
робота, що робиться змінною силою	[wə:k dʌn bai ðə ´vɛəriəbl ´fɔ:s]	work done by the variable force

13. Ряди

абсолютно збіжний	[əbsolut(ə)li kən´və:dʒiŋ]	absolutely converging
ведут себе	[bi´heiv]	behave
гармонічний ряд	[hær´mɔnik ´siəri:z]	harmonic series

довизначити	[ridi´fain]	redefine
збіжний ряд	[kən´və:(r)dʒiŋ ´siəri:z]	converging series
знакозмінний ряд	[ɔlter´neitiŋ ´siəri:z]	alternating series
знакопереміжний ряд	[ɔlter´neitiŋ in sain ´siəri:z]	alternating in sign series
мажорантний такий, що мажорується мажорируємий	[´meidʒərənt] [´meidʒəraizd]	majorant majorized
нескінченно спадаюча геометрична прогресія	[mɔnə´tounikəli dikri:ziŋ dʒi: ə'metrik prɔ´greʃn]	monotonically decreasing geometric progression
область збіжності	[dɔ´mein ɔv kən´və:dʒ(e)ns]	domain of convergence
ознака порівняння	[ə sain əv kɔm´pərisən]	a sign of comparison
причому	[bi´saidz]	besides
рівномірно збіжний	[´ju:nifɔ(r)mli kən´və:dʒiŋ]	uniformly converging
розбіжний ряд	[dai´və:(r)dʒiŋ ´siəri:z]	diverging series
розкладання	[iks´pænʃn]	expansion
степеневий ряд	[´pauə ´siəri:z]	power series
точка розриву	[´breik pɔint]	break point
умовно збіжний	[kʌn´diʃnəli kən´və:dʒiŋ]	conditionally converging
функціональний ряд	[´fʌnkʃnəl ´siəri:z]	functional series
часткова сума	[´pa:ʃəl sʌm]	partial sum
числовий ряд	[´nju:merik ´siəri:z]	numeric series

THEMATIC ENGLISH - UKRAINIAN DICTIONARY

1. Analytical geometry on the plane

altitude	[ǽltitju:d]	висота
angular factor	[ǽngjulə ´fæktə]	кутовий коефіцієнт
apex	[´eipeks]	вершина
area	[ɛəria]	площа
axis (axes)	[ǽksis (æksi:z)]	вісь (вісі) (геом.)
axle	[ǽksl]	вісь (фіз.)
curve	[kə:v]	крива
dropped	[dropt]	виходячий, проведений (з)
equalor then	[´i:kwələ(r) ðen]	більше ніж
equation	[ik´weiʃən]	рівняння
general	[´dʒenərəl]	загальний
grid	[grid]	сітка
inner angle	[´inə ǽngl]	внутрішній кут
onto	[əntu]	біля, по відношенню до
projection	[prə´dʒekʃən]	проекція
ratio	[´reiʃiəu]	відношення
scale	[skeil]	шкала, масштаб
value	[´vælju:]	значення

2. Elements of mathematic analysis

accept	[ək´sept]	приймати
add and deduct	[æd ənd di´dʌkt]	додати та відняти
aiming to	[´əimiŋ tu]	прямує
comparison	[kəm´pærisn]	порівняння
conclusion	[kən´klu:ʒn]	висновок

conjugate	[ˊkəndʒugeit]	спряжений, сполучений
consequence	[ˊkənsikwəns]	слідство, висновок
continuous	[kənˊtinjuəs]	неперервний
convenient	[kənˊvinjə:nt]	зручний
deployment	[diˊpləi:ment]	розкриття
doing	[du:iŋ]	роблячи
equivocations	[ikwivəˊkeiʃn]	невизначеність
fraction	[ˊfrækʃn]	дріб
identical conversions	[aidentikəl kənˊvə: ʃ(ə)n]	тотожні перетворення
indefinitely large	[inˊdefinitli la:dʒ]	нескінченно велика (н.в.)
inequality	[ˏini: ˊkwoliti]	нерівність
infinitesimal	[infinitəsiməl]	нескінченно мала (н.м.)
inverse	[inˊvə:s]	обернений
namely	[ˊneimli]	саме
particular careful	[ˏpəˊtikjuləˊkεəful]	обережно
passage to the limit	[ˊpæsidʒ tu ðə limit]	граничний перехід
picking out	[ˊpikiŋ aut]	виділити
remarkable	[riˊma:kəbl]	чудовий (мат.)
sequentially	[ˊsi:kwənʃiəli]	послідовно
sign	[sain]	знак
square trinomial	[skˊwεə traiˊnəumiəl]	квадратний тричлен
substitution	[ˏsʌbstiˊtju:ʃən]	заміна, підстановка
turning	[tə:niŋ]	обертаючийся
uncertainty	[ʌnˊsə:tenti]	невизначеність ($\frac{0}{0}$, $\frac{\infty}{\infty}$)
unchanged	[ʌnˊtʃeindʒd]	незмінний
x tend to 0	[iks tənd tə zirəu]	x прямує до 0 ($x \to 0$)

3. The fundamentals of the differential calculus for functions of one variable

achieves	[əˊtʃi:vs]	досягає
aciclic	[əˊsiklik]	неперіодична
aims to zero	[eimz tu ˊzir əu]	прямувати до нуля
among	[əˊmʌŋ]	серед
applied	[əˊplaid]	застосовується
approaches to	[əˊprəutʃiz tu]	прямувати до
below	[biˊləu]	нижче
border	[ˊbɔ:də]	кінець (обмеження)
concavity	[ˊkɔnˊkæviti]	угнутість
conclusion	[kənˊklu: ʒ(e)n]	висновок
convexity	[ˊkɔnˊveksiti]	опуклість
decreases	[dikˊri:s(iz)]	спадає
derivative	[diˊrivətiv]	похідна
disclosing	[disˊkləuziŋ]	розкриття
identity	[aiˊdentiti]	тотожність
immediately	[iˊmi:djətli]	одразу, водночас
implicit	[imˊplisit]	неявний
increases	[inˊkri:siz]	зростає
increment	[ˊinkrimənt]	приріст
inflection	[inˊflekʃn]	перегиб
initial	[iˊniʃ(ə)l]	початковий
instant	[ˊinstənt]	момент часу
instantaneous	[ˌinst(ə)nˊteinjəs]	моментальний
monotonicity	[mənəˊtənəsiti]	монотонність
neighborhood	[ˊneibəhud]	коло
nontrivial limit	[nɔnˊtriviəl limit]	не дорівнююча нулю границя
normal(line)	[ˊnɔ:m(ə)l]	нормаль

oddness	[´ɔdnes]	непарний
once again	[wʌns ə´gen]	ще раз
parity	[´pæriti]	парний
perpetuity	[ˌpə:pi´tju:iti]	нескінченність
relatively	[´relətivli]	відносно
require	[ri´kwaiə]	вимагати
requirement	[ri´kwaiəmənt]	вимога
restrict	[ris´trikt]	висновок
seeking	[si:kiŋ]	прямуючий
single-valued	[singl vælju:d]	однозначний
substitute	[´sʌbstitju:t]	заміна
sufficient	[sa´fiʃnt]	достатній
tangent(line)	[´tændʒ(ə)nt]	дотична
to create	[kri:´eit]	створити
unrestrictedly	[ˌʌnrist´rictidli]	необмежено
whence	[wens]	звідки

4. Indefinite Integral. Methods of integration

antiderivative	[əntide´rivətiv]	первісна
capital (upper) $F(x)$	[kæpitl]	велика літера $F(x)$
change of variable	[´tʃeindʒ ɔv ´veriəbl]	заміна змінної
complete square	[kəm´plit skwɛə]	повний квадрат
complex – conjugated (roots)	[´kʌmpleks - ´kɔndʒugeitid (ru:ts)]	комплексно-спряжені корені
conversion	[kʌn´və: ʃn]	звільнення
cosine	[kəu´sain]	косинус
cotangent	[´kəutændʒent]	котангенс
degree	[di´gri:]	ступінь
denominator	[dinɔminetə(r)]	знаменник

designated	[di´zaineitid]	наданий
direct	[dai´rəkt]	безпосередній, прямий
disclosure	[dis´klɔuʒə]	розклад
even number	[i:vən´nʌmba]	парне число
expand	[iks´pænd]	розкласти
exponent	[iks´ponənt]	показник
extract	[iks´trækt]	виділити
general	[´dʒenərəl]	універсальний
improper fraction	[imprɔpə´frækʃn]	неправильний дріб
in the power	[inði:´pauə]	у ступені
indefinite	[in´definit]	невизначений
integer	[´intidʒə]	ціле
integral part	[intəgrəl pa:t]	ціла частина
integrand	[´intəgrand]	підінтегральна функція
invariation	[inveri´eiʃən]	інваріантність, незмінність
linear algebraic equations	[´lainə ældʒi´breik i´kweiʃnz]	лінійні алгебраїчні рівняння
low $f(x)$	[lou]	мала літера $f(x)$
multiple	[´mʌltipl]	кратний
multiplier	[mʌlti´plaiə(r)]	множник
multiply	[´mʌltiplai]	помножити
nominator	[´nɔmineitə(r)]	чисельник
non-recurring	[nɔn-ri´kə:riŋ]	неповторний
odd number	[ɔd´nʌmba]	непарне число
polynomial	[polinɔmiəl]	поліном, многочлен
product	[´prɔdʌkt]	добуток
proper fraction	[´prɔpə´frækʃn]	правильний дріб

properties	[´prɔpətis]	властивості
rational fractions	[´ræʃən(ə)l frækʃn(z)]	раціональний дріб
real root	[riəl ru:t]	дійсний корінь
remainder	[ri´meində]	залишок
return integral	[ritə:n´intigr(ə)l]	зворотній інтеграл
similar	[´similə]	подібний
simplified	[simpli´faid]	спрощений
sine	[sain]	синус
subradical expression	[sʌb´rædikl iks´preʃn]	підкореневий вираз
substitution	[sʌbsti´tju:ʃn]	підстановка
summand	[sʌmænd]	доданок
tangent	[´tændʒənt]	тангенс
trigonometric substitution	[trigənəmetrik sʌbsti´tju: ʃn]	тригонометричні підстановки
undetermined coefficient	[ʌndi´tə:mind ˏkəui´fiʃənt]	невизначений коефіцієнт

5. Definite Integral

arc length	[a:(r)k lenθ]	довжина дуги
arc of cycloid	[a:(r)k ɔv´saiklɔid]	арка циклоїда
cross-section	[krɔs´ sәkʃ’n)]	поперечний переріз
curvilinear	[ˏkə:vi´liniə]	криволінійна
estimation	[esti´meiʃən]	оцінка
even function	[´i:vən ´fʌŋkʃən]	парна функція
improper	[im´prɔpə]	невласний
inertia moment	[inə:rʃia ´məument]	момент інерції
infinite break	[in´finit breik]	нескінченний розрив
odd function	[ɔd ´fʌŋkʃən]	непарна функція

quarter	[´kwə:tə]	чверть
rotation solid	[rəu´teiʃn ´sɔlid]	тіло обернення
smooth curve	[smu:θ kə:(r)v]	гладка крива
statical moment	[´stætik(ə)l ´məument]	статичний момент
that is confirmed by	[ðət iz kənfə:md bai]	що підтверджується
trapezoid	[træpizɔid]	трапеція
unbounded function	[Λn´baundid ´fΛŋkʃən]	необмежена функція

6. Elements of linear algebra

according to	[ə´kɔ:(r)din tu]	у відповідності до
algebraic cofactor	[ældʒi´breik kə´fæktə]	алгебраїчне доповнення
column	[´kɔləm]	стовбець
expand	[iks´pænd]	розкласти
linear transformation	[´liniə trænsfɔ:meiʃn]	лінійне перетворення
minor	[mainə]	мінор
row	[rou]	строка
unit matrix	[ju:nit ´meitriks]	одинична матриця
zero matrix	[´zirou ´meitriks]	нульова матриця

7. Elements of Vector Algebra

additional	[ædiʃn(ə)l]	сполучений
complanar, complinear	[´cʌmplenə cʌm´pliniə]	компланарний
direction	[di´rekʃn]	напрямок
marked as	[´ma:(r)kt əz]	позначається як, позначений
module	[´mɔdju:l]	модуль
opposite	[´ɔpəzit]	протилежний
orts	[ɔ(r)ts]	орти
polygonic	[pɔli´gɔnik]	многокутник

projection	[prəˊdʒekʃ(ə)n]	проекція
related	[riˊleitid]	зв’язаний
scalar	[ˊskæla:]	скалярний, скаляр
spatial point	[ˊspeiʃl ˊpɔint]	точка простору
the property of distribution	[ðəˊprɔpeti ɔv distriˊbju: ʃ n]	розподільча властивість
transferred	[trænsˊfə:d]	що перенесений

8. Analytical geometry in space

absolute term	[ˊæbsəlu:t tə:m]	вільний член
basis of the perpendicular	[ˊbeisis of pə:penˊdikjulə]	основа перпендикуляра
bundle of planes	[bʌndl ov pleins]	зв’язка площин
cone	[ˊkoun]	конгруентні відрізки
congruous intercepts	[ˊkɔŋgruəs intə:ˊsepts]	конус
cylinder	[ˊsilində]	циліндрична поверхня
cylindrical surface	[silindrikəl ˊsə:fis]	циліндр
ellipsoid	[ilipsɔid]	еліпсоїд
elliptic cylinder	[iˊliptik ˊsilində]	еліптичний циліндр
elliptic paraboloid	[iˊliptik peˊræbəlɔid]	еліптичний параболоїд
forming	[ˊfɔ:miŋ]	твірна
hyperbolic cylinder	[haipə:bəlik ˊsilində]	гіперболічний циліндр
hyperbolic paraboloid	[haipə:bəlik peˊræbəlɔid]	гіперболічний параболоїд
imaginary surface	[iˊmædʒin(ə)ri ˊsə:fis]	уявна поверхня
normal equation	[ˊnɔ:məl ikweiʃn]	нормальне рівняння
normalize multiplier	[ˊnɔ:məlaiz mʌltiplaiə(r)]	нормувальний множник
parabolic cylinder	[peræbolik ˊsilində]	параболічний циліндр

paraboloid	[pe´ræbəlɔid]	параболоїд
pencil of planes	[pensil ov pleins]	жмуток площин
single-shat hyperboloid	[´si ŋgl-ʃæt hai´pə:bəlɔid]	однопалий гіперболоїд
sphere	[sfiə]	сфера
surface of revolution	[s´ə:fis ov revə´lu: ʃn]	поверхня обертання
two-sheeted hyperboloid	[tu ´ʃi:tid hai´pə:bəlɔid]	двопалий гіперболоїд

9. Functions of Several Independent Variables

complex function	[kɔmpleks ´fʌnk ʃn]	складова функція
concentric spheres	[kʌn´sentrik ´sfiəz]	концентричні сфери
implicit function	[im´plisit ´fʌŋkʃn]	неявна функція
level lines	[´level ´lain(z)]	лінії рівня
level surfaces	[´level ´sə(r)fisiz]	поверхні рівня
mapping	[mæpiŋ]	відображення
nonempty set	[nʌn´empti set]	непуста множність
partial derivatives	[´pa:ʃ(ə)l de´ri:vətivz]	часткові похідні
stationary	[´steiʃn(ə)ri]	стаціонарний
the fastest	[ðə ´fastəst]	найбистріший
the greatest	[ðə greitəst]	найбільший
the least	[ðə li:st]	найменший

10. The Simple differential equations

characteristic equation	[´kærektəristik ikweiʃn]	характеристичне рівняння
complex-conjugate roots	[kɔmpleks kɔndʒugeit ru:ts]	комплексно-спряжені корені

dividing variables	[di´vaidiŋ ´vɛəriəblz]	відокремлювані змінні
homogenous equation	[hɔmə´dʒi:nəs ikweiʃn]	однорідне рівняння
inhomogenous equation	[inhɔmə´dʒi:nəs ikweiʃn]	неоднорідне рівняння
initial condition	[iniʃiəl kən´diʃn]	початкові умови
measurement	[,´meʒəment]	вимір
method of inspection	[meθɔd ɔv ins´pekʃn]	метод підбору
multiple root	[´mʌltipl ru:t]	кратний корінь
particular solution	[pa:(r)´tikju:lə sɔ´lu:ʃn]	часткове рішення
variational method of the random constant	[vɛəri´eiʃənl meθɔd ɔv ðə ´rændəm ´kɔnstənt]	метод варіації довільної сталої

11. Double integrals

closed domain	[´klouzd dɔ´mein]	замкнена область
curve-lined coordinates	[kə:v-laind kəu´ɔ:dinit(s)]	криволінійні координати
cylindrical surface	[silindrikəl ´sə:fis]	циліндрична поверхня
double integral	[´dʌbl ´integrəl]	подвійний інтеграл
geometric meaning	[dʒiɔ´metrik ´mi:niŋ]	геометричний зміст
gravity centre	[´græviti ´sent(r)ə]	центр ваги
inertia moment	[i´nə:rʃiə moumənt]	момент інерції
iterated integral	[itəreitid integrəl]	двократний інтеграл
plane plate	[plein pleit]	плоска пластівка
polar coordinates	[´pɔla:(r) kəu´ɔ:dinit(s)]	полярні координати

repeated integral	[ripi:tid ´integrəl]	повторний інтеграл
single-valied correspondence	[siŋgl-´vælid ,kɔris´pɔndəns]	взаємно-однозначна відповідність
static moment	[stə´tik moumənt]	статичний момент

12. Contour integrals

arc length	[a(r)k lenθ]	довжина дуги
astroid	[´æstrɔid]	астроїда
at detour along the contour *K*	[ət də´tuə ə´lɔŋ ðə ´kɔntuə *K*]	при пробігу контура *K*
broken line, break line	[´broukn lain]	ломана лінія
contour integrals of the I st kind	[´kɔntuə ´integrəlz ɔv ðə ´fə:st ´kaind]	криволінійний інтеграл і роду
contour integrals of the II st kind	[´kɔntuə ´integrəlz ɔv ðə ´sekənd ´kaind]	криволінійний інтеграл її роду
curved way	[kə:vd wei]	криволінійний путь
elementary arc	[eli´mentəri a´(r)k]	елементарна дуга
integration contour	[intəg´reiʃn ´kɔntuə]	контур інтегрування
one valued function	[van vælju:d ´fʌnkʃn]	однозначна функція
piecewise smooth spatial curve	[,pi:s´waiz ´smu:θ ´speiʃl kə:v]	кусково-гладка просторова крива
simply connected domain	[simpli kə´nəktid dɔ´mein]	однозв’язна область
smooth curve	[smu:θ kə:v]	гладка крива
variable density	[vɛəriəbl densiti]	змінна густина
work done by the variable force	[wə:k dʌn bai ðə ´vɛəriəbl ´fɔ:s]	робота, що робиться змінною силою

13. Series

a sign of comparison	[ə sain əv kɔm´pərisən]	ознака порівняння
absolutely converging	[əbsolut(ə)li kən´və:dʒiŋ]	абсолютно збіжний
alternating in sign series	[ɔlter´neitiŋ in sain ´siəri:z]	знакопереміжний ряд
alternating series	[ɔlter´neitiŋ ´siəri:z]	знакозмінний ряд
behave	[bi´heiv]	ведут себе
besides	[bi´saidz]	причому
break point	[´breik pɔint]	точка розриву
conditionally converging	[kʌn´diʃnəli kən´və:dʒiŋ]	умовно збіжний
converging series	[kən´və:(r)dʒiŋ ´siəri:z]	збіжний ряд
diverging series	[dai´və:(r)dʒiŋ ´siəri:z]	розбіжний ряд
domain of convergence	[dɔ´mein ɔv kən´və:dʒ(e)ns]	область збіжності
expansion	[iks´pænʃn]	розкладання
functional series	[´fʌnkʃnəl ´siəri:z]	функціональний ряд
harmonic series	[hær´mɔnik ´siəri:z]	гармонічний ряд
majorant	[´meidʒərənt]	мажорантний
majorized	[´meidʒəraizd]	такий, що мажорується мажорируємий
monotonically decreasing geometric progression	[mɔnə´tounikəli dikri:ziŋ dʒi: ə'metrik prɔ´greʃn]	нескінченно спадаюча геометрична прогресія
numeric series	[´nju:merik ´siəri:z]	числовий ряд
partial sum	[´pa:ʃəl sʌm]	часткова сума
power series	[´pauə ´siəri:z]	степеневий ряд
redefine	[ridi´fain]	довизначити
uniformly converging	[´ju:nifɔ(r)mli kən´və:dʒiŋ]	рівномірно збіжний

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Навчальне видання

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