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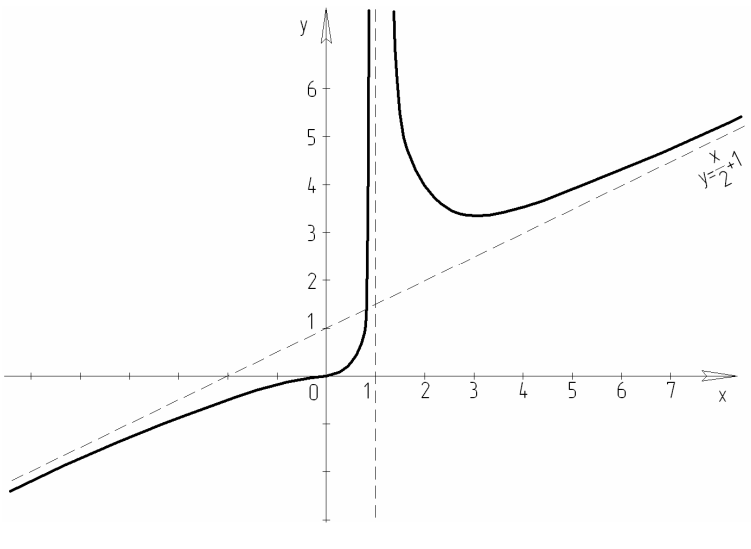
O.V. BORAKOVSKIY, O.I. ROPAVKA

ПОСІБНИК ДЛЯ РОЗВ'ЯЗАННЯ ЗАДАЧ З ВИЩОЇ МАТЕМАТИКИ

HANDBOOK FOR PROBLEM SOLVING IN HIGHER MATHEMATICS



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Рекомендовано Міністерством освіти і науки України як навчальний посібник для студентів вищих навчальних закладів

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О.В.Бораковський, О.І.Ропавка
O.V.Borakovskiy, O.I.Ropavka

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У посібнику наведено стислі теоретичні відомості та їх практичне застосування до розв'язання задач з аналітичної геометрії на площині, елементів математичного аналізу, основ диференціального та інтегрального числення, елементів лінійної та векторної алгебри, функцій декількох змінних, звичайних диференціальних рівнянь, подвійних та криволінійних інтегралів, рядів. Розглянуто методи розв'язання типових задач. Надані завдання для самостійної роботи. Посібник призначено для студентів вищих навчальних закладів для ознайомлення та практичного застосування методів вищої математики англійською мовою.

This handbook contains a short theoretical material and its practical application while problem solving in Analytical geometry on the plane and in space, on elements of Mathematic Analysis, on the basis of differential and Integral calculations, on elements of linear and vector algebra, on functions with some variables and simple differential equations double and contour integrals, series. Methods of typical problem solutions are considered and self-study tasks are given as well. It is intended for the students of higher educational institutions to get theoretical knowledge and practical skills of higher mathematics methods application.

Рецензенти:

Ю.Є.Обжерін, доктор технічних наук, професор, завідувач кафедри вищої математики (Севастопольський національний технічний університет);

А.Д.Тевяшев, доктор технічних наук, професор, завідувач кафедри прикладної математики (Харківський національний технічний університет радіоелектроніки);

О.М.Шупіков, доктор технічних наук, професор кафедри вищої математики (Харківський національний економічний університет).

ЗМІСТ

CONTENT

ПЕРЕДМОВА

В основу посібника покладено програми вищої математики для студентів факультетів інженерної екології міст і містобудівельного Харківської національної академії міського господарства (ХНАМГ).

Метою посібника є практичне застосування математичного апарату до розв’язання задач з аналітичної геометрії на площині, елементів математичного аналізу, основ диференціального числення, невизначених та визначених інтегралів, елементів лінійної та векторної алгебри, функцій декількох змінних, звичайних диференціальних рівнянь, подвійних та криволінійних інтегралів, рядів.

Викладення тексту англійською мовою особливого значення набуває у рамках Болонського процесу: дає можливість ознайомитись з основами перекладу математичних текстів та застосувати отримані знання на практиці.

Для зручності засвоєння матеріалу наприкінці посібника наведені тематичні англо-український та українсько-англійський словники.

Достатня кількість розв’язаних типових прикладів та завдань для самостійної роботи дає змогу студентам самостійно опанувати даний курс вищої математики, поповнити словниковий запас з англійської мови та підготуватися до складання іспитів.

Посібник складено на основі курсів лекцій, що читалися авторами на факультетах інженерної екології міст і містобудівельному ХНАМГ зі збереженням стилістики та позначень, що суттєво полегшує засвоєння матеріалу англійською мовою.

Зауваження та пропозиції надсилайте на кафедру вищої математики ХНАМГ за адресою: 61002, м. Харків, вул. Революції, 12.

INTRODUCTION

To the basis of the following handbook was laid the syllabus in Higher Mathematics for students of City Engineering Ecology Faculty and Town-Planning Department of Kharkiv National Municipal Academy (KNMA).

The handbook is aimed at practical application of mathematic apparatus to the problem solutions in analytical geometry on the plane and in space, elements of mathematic analysis on the basis of Differential Calculations indefinite and definite Integrals, elements of linear and Vector Algebra, functions with some variables, simple differential equations, double and contour integrals, series.

Being translated into English, the following handbook acquires special importance in the frames of the Bologna Process due to this edition they'll be acquainted with the basis of the translation of technical English and its practical application.

To make adoption of the material easier each unit is followed by the list of key words in the frames of study translated into Russian.

The sufficient number of solved typical examples and multiple choice home tasks gives the opportunity for students to master the given syllabus of Higher Mathematics self-studying and to memorize lexis. The edition was written on the basis of the lecture course, given by the author of the faculty of city Engineering Ecology Department with stylistics and Symbols being kept, that makes the technical translation significantly easier.

1. ANALYTICAL GEOMETRY

1.1. The straight line on the plane

In the early XVII century, the French mathematician Rene Descartes introduced the idea of a grid or system for location and plotting points on the plane.

Onto this grid, called a Cartesian graph two straights are drawn at right angle to one another, called the rectangular axes, we must determine:

1) starting point or origin 0;

2) the positive direction of axes;

3) unit of dimension or scale.

The position of any point (say, A) can then be described by two numbers: one referring to the horizontal axis X and one to the vertical axis Y. The two numbers are called the Cartesian coordinates, after Descartes.

The straight line is a set of points which coordinates in some system of the Cartesian coordinates satisfy the following equation:

$$Ax + By + C = 0, \quad \text{where } A^2 + B^2 \neq 0.$$

This is a general equation of the straight line.

An equation of the form $y = kx + b$ can by represented graphically by a straight line – equation with an angular factor $k = tg\alpha$, where α is an angle of lean of the given line to the axis OX or X.

If a straight line $Ax + By + C = 0$ doesn't cross the beginning of coordinates, i.d. $C \neq 0$ we can obtain straight line equation in the intercepts $\frac{x}{a} + \frac{y}{b} = 1$ where $a = -\frac{C}{A}$ $b = -\frac{B}{A}$ - are intercepts cut by the straight line on the coordinate axes.

Having multiplied straight line equation $Ax + By + C = 0$ by normalize multiplier $\mu = \pm\frac{1}{\sqrt{A^2 + B^2}}$, which sign is opposite to the absolute term C sign, we are obtaining a normal straight line equation,

$$cos\,\varphi + y\,sin\,\varphi - p = 0$$

where p – is the length of perpendicular dropped from the beginning of coordinates to the straight line, φ – an angle obtained from this perpendicular with the positive directions of OX axis.

Pointed angle between two straight lines $y = k_1 x + b_1$ and $y = k_2 x + b_2$ is calculated with the help of the formula

$$tg\alpha = \left| \frac{k_2 - k_1}{1 + k_1 k_2} \right|.$$

Then the condition of two straight lines parallelity is: $k_2 = k_1$ the condition of perpendicularity is: $1 + k_1 k_2 = 0$ or $k_1 = -\frac{1}{k_2}$.

The straight line equation going through $M(x_1; y_1)$ point with the given angle coefficient k is presented as follows:

$$y - y_1 = k(x - x_1).$$

The straight line equation going through two points $M_1(x_1; y_1)$ and $M_2(x_2; y_2)$ is presented as follows:

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}.$$

An angle coefficient of this straight line is calculated according to the formula $k = \frac{y_2 - y_1}{x_2 - x_1}$.

If $x_2 = x_1$, then straight line equation is presented as $x = x_1$.

If $y_2 = y_1$, then the straight line equation is presented as follows $y = y_1$.

Distance d from point $M(x_0; y_0)$ to the straight line $Ax + By + C = 0$ is calculated according to the formula:

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

Equation of angle bisectors between the straight lines $A_1 x + B_1 y + C_1 = 0$ and $A_2 x + B_2 y + C_2 = 0$ is presented as follows:

$$\frac{A_1 x + B_1 y + C_1}{\sqrt{A_1^2 + B_1^2}} \pm \frac{A_2 x + B_2 y + C_2}{\sqrt{A_2^2 + B_2^2}} = 0$$

Equation of straight lines pencil go through the intersection point of the given straight lines is presented as follows:

$$A_1 x + B_1 y + C_1 + \lambda(A_2 x + B_2 y + C_2) = 0,$$

where λ is – a number multiplier

Example. Draw a graph to represent the line *AB:* $y = 2x - 4$ for values of x from –1 to 3.

Solution. *y*(–1) = –2 – 4 = –6, *y*(–3) = 6 – 4 = 2. Then *A*(–1, –6), *B*(3, 2).

Example. Find projection of the point $C(2;3)$ on the line passing through the points $A(3;0)$ and $B(-3;3)$.

Solution. Equation of the line passing throuth the point A and B is AB:

$$\frac{y-y_A}{y_B-y_A}=\frac{x-x_A}{x_B-x_A};\ \frac{y-0}{3-0}=\frac{x-3}{-3-3};\ y=-\frac{1}{2}x+\frac{3}{2};\ k_{AB}=-\frac{1}{2}.$$

Then angular factor of the line CN is $k_{CN}=+2$, as $k_{CN}=-\frac{1}{k_{AB}}$ from condition of perpendicularity.

Equation of the line passing through the point C with an angular factor k is $y-y_C=k_{CN}(x-x_C)$; $y-3=2(x-2)$ or $y=2x-1$.

Solving together : $\begin{cases} y=-\frac{x}{2}+\frac{3}{2} \\ y-2x-1 \end{cases}$ we find coordinates of the point $N(1;1)$.

Projection of the point C on the line AB is the point $N(1;1)$.

Example. We are given coordinates apexes of a triangle ABC $A(-2;-2)$; $B(4;1)$; $C(0;4)$ (Fig.1). Using methods of the analytical geometry do the following:

1) Find the distance between point A and point B;
2) form equation of the sides AB, AC;
3) form equation of the altitude dropped from the apex C;
4) find area of the ΔABC;
5) find the inner angle of the triangle at the apex A; (α);
6) calculate length of the altitude dropped from the apex C;
7) form equation of the straight line passing through the center of gravity $(\bar{x};y)$ the ΔABC and parallel to the side AC;
8) form equation of the median dropped from the apex C;
9) draw ΔABC.

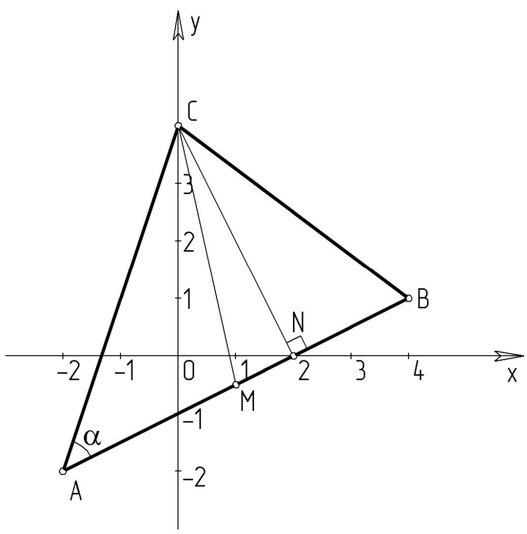


Fig.1

1) $d_{AB}=\sqrt{(x_B-x_A)^2+(y_B-y_A)^2}=\sqrt{(4+2)^2+(1+2)^2}=\sqrt{36+9}=3\sqrt{5}$;

2) $AB: \dfrac{y-y_A}{y_B-y_A}=\dfrac{x-x_A}{x_B-x_A}$; $\quad \dfrac{y+2}{1+2}=\dfrac{x+2}{4+2}$;

$x-2y-2=0$; $\quad y=\dfrac{1}{2}x-1$; $\quad k_{AB}=\dfrac{1}{2}$;

$AC:\ y=3x+4$; $\quad k_{AC}=\dfrac{y_C-y_A}{x_C-x_A}=\dfrac{4+2}{0+2}=3$;

3) $y-y_C=-\dfrac{1}{k_{AB}}(x-x_C)$;

$CN:\ k_{CN}=-\dfrac{1}{k_{AB}}$; $\quad y-4=-\dfrac{1}{1/2}(x-0)$; $\quad y=-2x+4$;

4) $S_{\Delta ABC}=\dfrac{1}{2}\left\|\begin{matrix} x_A & y_A \\ x_B & y_B \\ x_C & y_C \\ x_A & y_A \end{matrix}\right\|$, look through the lectures

$S_{\Delta ABC}=\dfrac{1}{2}\left\|\begin{matrix} -2 & -2 \\ 4 & 1 \\ 0 & 4 \\ -2 & -2 \end{matrix}\right\|=\dfrac{1}{2}\left|-2+16+0+8-0+8\right|=\dfrac{1}{2}30=15$;

5) $tg\alpha=\left|\dfrac{k_{AC}-k_{AB}}{1+k_{AC}k_{AB}}\right|=\left|\dfrac{3-\frac{1}{2}}{1+\frac{3}{2}}\right|=1$; $\quad \alpha=arctg1=\dfrac{\pi}{4}$;

6) $d_{CN}=\dfrac{\left|1\cdot x_C-2y_C-2\right|}{\sqrt{1^2+(-2)^2}}=\dfrac{10}{\sqrt{5}}$;

$S_{\Delta ABC}=\dfrac{1}{2}\left|AB\right|d_{CN}=\dfrac{1}{2}3\sqrt{5}\,\dfrac{10}{\sqrt{5}}=15$, as a check;

7) $\bar{x}=\dfrac{x_A+x_B+x_C}{3}=\dfrac{2}{3}$;

$\bar{y}=\dfrac{y_A+y_B+y_C}{3}=1$;

$$y-\bar{y}=k_{AC}(x-\bar{x}); \quad y-1=3(x-\frac{2}{3}); \quad y=3x-1;$$

8) find the middle of the section AB – point M :

$$x_M=\frac{x_A+x_B}{2}=\frac{-2+4}{2}=1; \quad y_M=\frac{y_A+y_B}{2}=\frac{-2+1}{2}=-\frac{1}{2};$$

$$CM: \ \frac{y-y_M}{y_C-y_M}=\frac{x-x_M}{x_C-x_M}; \quad \frac{y+\frac{1}{2}}{4+\frac{1}{2}}=\frac{x-1}{0-1};$$

$$-y-\frac{1}{2}=\frac{9}{2}x-\frac{9}{2}; \quad y=-\frac{9}{2}x+4.$$

1.2. Curves of the second order

Curve of the second order on a plane is called a set of points which coordinates are of the same system as the Cartesian coordinates satisfy the following equation

$$Ax^2+2Bxy+Cy^2+2Dx+2Ey+F=0, \text{ and}$$

$$A^2+B^2+C^2\neq 0.$$

Let's consider elementary (canonical) equations of curves of the second order.

1. If $A=C$, and $B=0$, we have an equation of circle.

Circle is a geometrical place of points equidistant from a certain point called centre.

$$x^2+y^2=R^2.$$

If a coordinate of centre $O_1(a,b)$, then we have

$$(x-a)^2+(y-b)^2=r^2;$$

$$x^2-2ax+a^2+y^2-2by+b^2-r^2=0;$$

$$x^2+y^2+lx+my+n=0;$$

$$l=-2a; \ m=-2b; \ n=a^2+b^2-r^2;$$

if $l^2+m^2-4n>0$, we have an equation of real circle;

if $l^2+m^2-4n=0$, we have a point $O_1\left(-\frac{l}{2};-\frac{m}{2}\right)$ or a circle with radius equal zero

if $l^2+m^2-4n<0$, we have an imaginary circle.

Example. Find coordinates of centre O_1 and radius (r) of circle

a) $x^2-2x+y^2+4y+1=0$

b) $x^2+3x+y^2+y+2=0$

Solution. We must extract the whole square: $(a\pm b)^2=a^2\pm 2ab+b^2$;

a) $\underbrace{x^2-2x+1}-1+\underbrace{y^2+4y+4}-4+1=0$

$(x-1)^2+(y+2)^2=2^2$; $O_1(1;-2)$; $r=2$;

b) $\underbrace{x^2-2\cdot\frac{2}{3}x+\frac{9}{4}}-\frac{9}{4}+\underbrace{y^2+2\cdot\frac{1}{2}y+\frac{1}{4}}-\frac{1}{4}+\frac{8}{4}=0$

$(x+\frac{3}{2})^2+(y+\frac{1}{2})^2=\left(\frac{\sqrt{2}}{2}\right)^2$; $O_1(-\frac{3}{2};-\frac{1}{2})$; $r=\frac{\sqrt{2}}{2}$.

2. Ellipse is a geometrical place of the points which sum of distances to two certain points called focuses is constant (usually considered equal to $2a$) equalor than the distance between focuses. Let's introduce the Cartesian coordinate system thus that the axis OX will pass through the focuses F_1F_2 and the axis OY will divide the interval F_1F_2 in halves, then the equation is:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ (Fig.2)},$$

called a canonical equation of an ellipse. The numbers $2a$ and $2b$ represent the length of major and smaller axles of an ellipse.

They are connected by the following equality: $c^2=a^2-b^2$.

The points $(a,0)$; $(-a,0)$; $(0,b)$; $(0,-b)$ are called apexes of an ellipse.

Ratio of half of focal distance to the length of half a major axle is called eccentricity of an ellipse $e=\frac{c}{a}$.

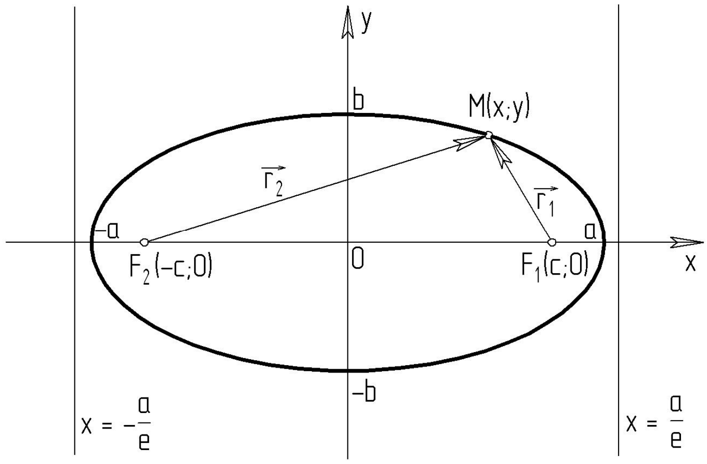


Fig.2

Eccentricity of an ellipse features the level of elongation of an ellipse, and for ellipse $e < 1$. For a circle $e = 0$

$$r_1 + r_2 = 2a > 2c$$

$$\frac{r_1}{d_1} = \frac{r_2}{d_2} = e < 1.$$

Hyperbola is a geometrical place of points which absolute difference of distances to two certain points called focuses is constant (usually considered equal to $2a$) smaller than the distance between focuses (Fig.3).

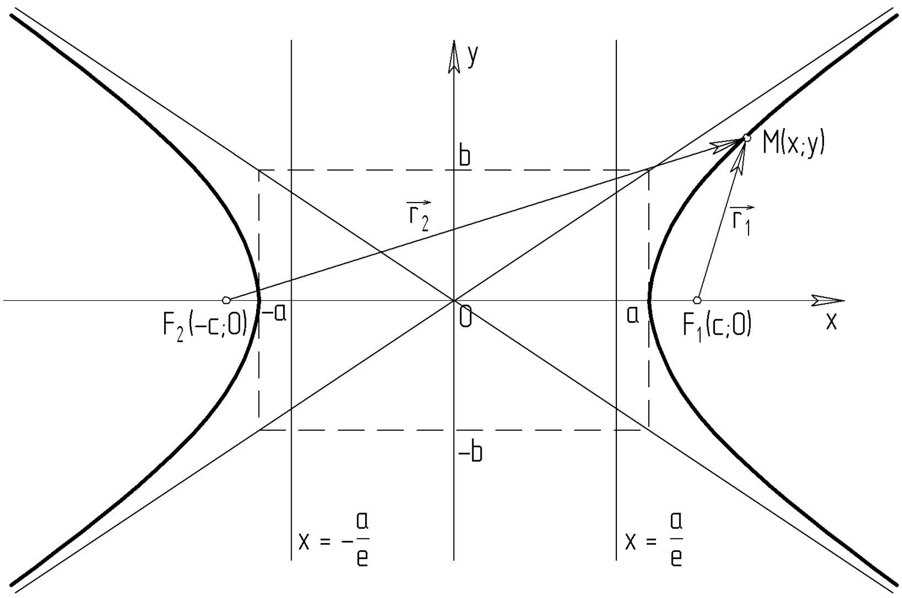


Fig.3

If we place the Cartesian coordinate system the same way as in the previous case then we obtain canonical equation of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

The numbers $2a$ and $2b$ represent the length of real and imaginary axles of hyperbola.

They are connected by the following equality: $c^2=a^2+b^2$.

The point $(a,0)$ and $(-a,0)$ are called apexes of hyperbola. The ratio of half of focal distance to the length of half real axle is called eccentricity of hyperbola $e=\frac{c}{a}$. It's obvious that for hyperbola $e>1$.

The lines $y=\pm\frac{b}{a}x$ are called asymptotes of hyperbola. If $a=b$, then hyperbola is called rectangular.

Parabola is a geometrical place of points equidistant from one certain point called focus and a certain line called directrix. Let's choose the system of coordinates as it is shown on the Fig.4. Then canonical equation of a parabola is as follows $y^2=2px$, where p is a parameter of parabola, numerically equal to the distance between focus and directrix. $r=d$; $r=x+\frac{p}{2}$.

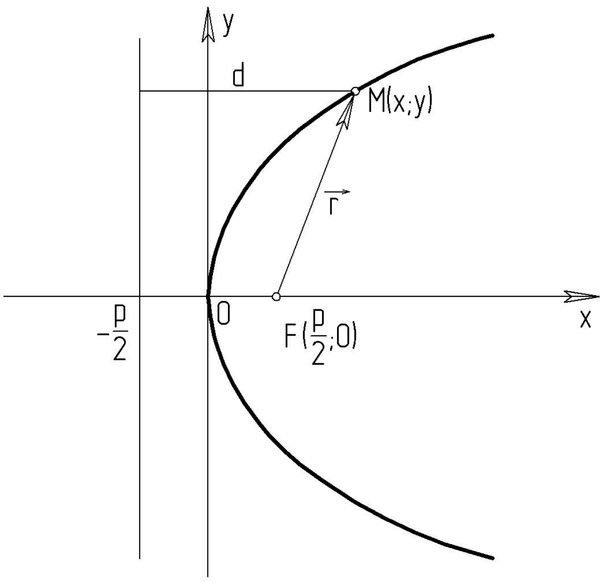


Fig.4

We should mention that the concept of directrix is also defined for ellipse and hyperbola, but these curves have two directrixes which equations make $x=\pm\frac{a}{e}$.

These curves are perpendicular to focal axes and located outside of apexes in case of ellipse ($e < 1 \Rightarrow |x| > a$) and between apexes in case of hyperbola ($e > 1 \Rightarrow |x| < a$). Thus the right one is considered the directrix correspondent to the right focus of a curve, and the left one is accordingly correspondent to the left focus.

There takes place the so called focal – directorial property of curves of the second order: the ratio of distance between any point $M(x;y)$ of a curve and focus directrix (d_M) is constant equal to eccentricity: $\frac{r_M}{d_M} = e$. Eccentricity of a parabola as it follows from its definition is equal to 1.

Example. a) Point $A(1;2)$ lies on the parabola and OX is the axis of symmetry;

b) equation of directrix of parabola is $y = -2$. Form equation of these parabolies.

Solution.

a) $y^2 = 2px$; $2^2 = 2p \cdot 1$; $p = 2$; $y^2 = 4x$;

b) $y = -2$; $-\frac{p}{2} = -2$; $p = 4$; $x^2 = 2py$; $x^2 = 8y$.

Example. a) Point $A(0;-2)$ and $B(\frac{\sqrt{15}}{2};1)$ lie on the ellipse.

b) Point $A(0;\sqrt{11})$ is an apex and $e = \frac{5}{6}$

Form equations of these ellipses.

Solution. a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$;

$$\left.\begin{aligned} A:&\ \frac{0}{a^2} + \frac{(-2)^2}{b^2} = 1;\quad b^2 = 4; \\ B:&\ \frac{15}{4a^2} + \frac{1}{b^2} = 1; \end{aligned}\right\} \text{Solving together,}$$

find a^2 and b^2, $\frac{15}{4a^2} + \frac{1}{4} = \frac{4}{4}$; $a^2 = 5$;

$$\frac{x^2}{5} + \frac{y^2}{4} = 1; \quad \frac{x^2}{(\sqrt{5})^2} + \frac{y^2}{(2)^2} = 1.$$

b) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; $\frac{0}{a^2}+\frac{11}{b^2}=1$; $b^2=11$

$e=\frac{c}{a}=\frac{5}{6}$; $c^2=a^2-b^2$; $\frac{a^2-b^2}{a^2}=\frac{25}{36}$; $\frac{36}{36}-\frac{11}{a^2}=\frac{25}{36}$;

$\frac{11}{36}=\frac{11}{a^2}$; $a^2=36$; $\frac{x^2}{36}+\frac{y^2}{11}=1$; $\frac{x^2}{6^2}+\frac{y^2}{(\sqrt{11})^2}=1$.

Example. a) Point $A(3;4)$ and $B(5;4\sqrt{5})$ lie on the hyperbola;

b) Point $A(6;0)$ is an apex and $k=\frac{\sqrt{2}}{2}$. Form equations of these hyperbolies.

Solution. a) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$;

$$\left.\begin{aligned} A:&\ \frac{9}{a^2}-\frac{16}{b^2}=1 \\ B:&\ \frac{25}{a^2}-\frac{80}{b^2}=1 \end{aligned}\right|\ \begin{matrix}\mathbf{-5}\\ \mathbf{+}\end{matrix}$$

$\frac{-45}{a^2}+\frac{25}{a^2}+\frac{80}{b^2}-\frac{80}{b^2}=-5+1$; $\frac{-20}{a^2}=-4$; $a^2=5$;

$\frac{25}{5}-\frac{80}{b^2}=1$; $\frac{80}{b^2}=4$; $b^2=20$; $\frac{x^2}{5}-\frac{y^2}{20}=1$; $\frac{x^2}{(\sqrt{5})^2}-\frac{y^2}{(2\sqrt{5})^2}=1$;

b) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$; $a=6$; $k=b/a=\sqrt{2}/2$;

$b=\frac{\sqrt{2}}{2}a=\frac{\sqrt{2}}{2}\cdot 6=3\sqrt{2}$; $\frac{x^2}{6^2}-\frac{y^2}{(3\sqrt{2})^2}=1$.

Home task

1. We have coordinates apexes of a triangle ABC: $A(-1;-6)$; $B(-3;5)$; $C(4;0)$.
2. Using methods of the analytical geometry do 1) $\div$9) for example p. 11.
3. Find coordinates of centre and radius of circle:
4. a) $x^2-5x+y^2+7y=0$

b) $2x^2 + x + 2y^2 - 3y + 1 = 0$

5. a) Point $A(-3;0)$ and $B\left(1,\frac{\sqrt{40}}{3}\right)$ lie on the ellipse

b) Point $A\left(0;-\sqrt{11}\right)$ is an apex and $e = \frac{5}{6}$

Form equations of these ellipses

6. a) Point $A\left(\sqrt{6};0\right)$ and $B\left(-2\sqrt{2};1\right)$ lies on the hyperbola

b) $R\frac{\sqrt{17}}{8}$; $C = 9$. Form equations of these hyperbolies.

7. Equation of directrix of parabola is $x = \frac{1}{2}$. Form equation of this parabola.

2. ELEMENTS OF MATHEMATIC ANALYSIS

2.1. Limit of a numeric sequence and function

Definition. The number a is called the limit of the numeric $\{x_n\}$ if for any given positive number ε we can present ($\exists$) such number $N(\varepsilon)$ that all values x_n, $\forall n \geq N$, will satisfy the inequality $|x_n - a| < \varepsilon$. $\lim\limits_{n\to\infty} x_n = a$.

The number A is called the limit of the function $y = f(x)$ at $x \to a$, if for any positive number ε there exists such positive number $\delta > 0$ that $\forall x$ satisfying the inequality $|x - a| < \delta$ the inequality $|f(x) - A| < \varepsilon$ is true $\lim\limits_{x\to a} f(x) = A$.

I remarkable limit: $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$.

II remarkable limit: $\lim\limits_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = \lim\limits_{\alpha\to 0}(1+\alpha)^{1/\alpha} = e$, there e=2,71828…

2.2. Comparison of two infinitesimal values

The value $\alpha(x)$, at $x \to a$ is called infinitesimal if $\lim\limits_{x\to a}\alpha(x) = 0$.

Let the variables $\alpha(x)$ and $\beta(x)$, be infinitesimal values, i.e. $\lim\limits_{x\to a}\alpha(x) = \lim\limits_{x\to a}\beta(x) = 0$.

Then:

$$\lim_{x\to a}\frac{\alpha(x)}{\beta(x)}=\begin{cases}0, & \alpha(x) \text{ is infinitesimal of the higher order then } \beta(x)\text{: } \alpha(x)=o(\beta(x));\\ 1, & \alpha(x) \text{ and } \beta(x) \text{ are called equivallent infinitesimal: } \alpha(x)\sim\beta(x);\\ m\neq 0;\ 1, & \alpha(x) \text{ and } \beta(x) \text{ are called infinitesimal of the same order;}\\ \infty, & \beta(x) \text{ is infinitesimalof the higher order then } \alpha(x)\text{: } \beta(x)=o(\alpha(x)).\end{cases}$$

The next infinitesimal are equivalent at x tend to 0 ($x\to 0$): $x\sim\sin x$; $x\sim \mathrm{tg}x$; $x\sim\arcsin x$; $x\sim \mathrm{arctg}x$; $x\sim\ln(1+x)$; $x\sim e^x-1$.

2.3. Techniques of the limits calculus

While calculating the limits of functions, the rule of passage to the limit under the sign of continuous function is used. This rule is formulated as follows:

$$\lim_{x\to a} f(x)=f(\lim_{x\to a} x).$$

All elementary functions are continuous in their range of definition.

While calculating limits first of all we are to substitute argument of the function by its limiting value and find out whether there is uncertainty.

To *indefinite* expressions relay the following:

$$\frac{0}{0},\ \frac{\infty}{\infty},\ \infty*0,\ \ \infty-\infty,\ \infty^0,\ 0^0,\ 1^\infty.$$

If after substitution of the limiting value of an argument we obtain an *indefinite expression* then we are to carry out some identical conversions that will *eliminate the obtained uncertainty* and then calculate the needed limit.

Let's sequentially consider the standard cases of deployment of the indefinite expressions (equivocations).

Calculus of the limit of a rational function

$\lim\limits_{x\to x_0}\dfrac{P_n(x)}{Q_m(x)}$, where $P_n(x)$, $Q_m(x)$ are polynomials of the orders n and m, $n\in N$, $m\in N$.

1. $\lim\limits_{x\to x_0}\dfrac{P_n(x)}{Q_m(x)}=\dfrac{0}{0}$

Let's carry out identical conversions in order to eliminate uncertainty, i.e. picking out in numerator and denominator the factor aiming to 0, and namely ($x-x_0$).

In this case we are to remember the following:

1) Consequence of the *theorem by Bezout*: if x_0 is root of the polynomial $P_n(x)$, i.e. $P_n(x_0)=0$, then $P_n(x)$ is divided on the binomial $(x-x_0)$ without the remainder:

$$P_n(x)=(x-x_0)R_{n-1}(x);$$

2) The square trinomial $P_2(x)=ax^2+bx+c$ at which $D\ge 0$ ($D=b^2-4ac$), can by presented as the product of linear factors: $ax^2+bx+c=a(x-x_1)(x-x_2)$, where x_1 and x_2 are roots of the square trinomial.

Example. $\lim\limits_{x\to -1}\dfrac{2x^2-3x-4}{x^3+4x+4}=\left\|\dfrac{2+3-4}{-1-4+4}\right\|=-1$.

In this case the result is obtained at once as substitution $x\to -1$ gives ratio of two numbers with denominator $\ne 0$.

Example. $\lim\limits_{x\to 1}\dfrac{x^3-2x^2+5x-4}{x^3-3x^2+3}=\left\|\dfrac{1-2+5-4}{1-3+3}=\dfrac{0}{1}\right\|=0$.

Example. $\lim\limits_{x\to 0}\dfrac{(1+x)^2+(1+2x)^2}{2x+x^3}=\left\|\dfrac{1+1}{0}\right\|=\infty$.

Example. $\lim\limits_{x\to 1}\dfrac{x^3-1}{2x^3-x^2-1}=\left\|\dfrac{0}{0}\right\|$.

In this case we are to eliminate uncertainty, i.e. pick out in numerator and denominator the factor aiming to zero. To do this let's present the numerator as $x^3-1=(x-1)(x^2+x+1)$. The denominator according to the theorem by Bezout is divided on $x-1$ without the remainder:

$$\begin{array}{rr|l}
-\ & 2x^3-x^2-1 & x-1 \\ \cline{3-3}
 & \underline{2x^3-2x^2} & 2x^2+x+1 \\
-\ & x^2-1 & \\
 & \underline{x^2-x} & \\
-\ & x-1 & \\
 & \underline{x-1} & \\
 & 0 &
\end{array}$$

Now the denominator can by presented as $2x^3-x^2-1=(x-1)(2x^2+x+1)$. Thus we finish calculus of the given limit, i.e.

$$\lim_{x\to 1}\frac{(x-1)(x^2+x+1)}{(x-1)(2x^2+x+1)}=\lim_{x\to 1}\frac{(x^2+x+1)}{2x^2+x+1}=\left\|\frac{1+1+1}{2+1+1}\right\|=\frac{3}{4}.$$

Example. $\lim\limits_{x\to\infty}\dfrac{2x^2-5x+1}{x^2+2x^2-1}=\left\|\dfrac{\infty}{\infty}\right\|=\lim\limits_{x\to\infty}\dfrac{2\dfrac{x^2}{x^2}-5\dfrac{x}{x^2}+\dfrac{1}{x^2}}{\dfrac{x^2}{x^2}+2\dfrac{x}{x^2}-\dfrac{1}{x^2}}=\dfrac{2}{1}=2$.

Example. $\lim\limits_{x\to\infty}\dfrac{x^4+3x^2+1}{x^3-x^2+3x}=\left\|\dfrac{\infty}{\infty}\right\|=\lim\limits_{x\to\infty}\dfrac{x^4}{x^3}=\lim\limits_{x\to\infty}\dfrac{x}{1}=\infty \quad \begin{pmatrix} n=4 \\ m=3,\ n>m \end{pmatrix}$.

Example. $\lim\limits_{x\to\infty}\dfrac{x^2-7x+10}{x^3-2x^2+5}=\left\|\dfrac{\infty}{\infty}\right\|=\lim\limits_{x\to\infty}\dfrac{x^2}{x^3}=\lim\limits_{x\to\infty}\dfrac{1}{x}=0 \quad \begin{pmatrix} n=2 \\ m=3,\ n<m \end{pmatrix}$.

Calculus of the limits of functions with irrational expressions

While calculating the limits having irrational expression turning to zero at $x\to a$ we are to pick the factor $(x-a)\to 0$. We can do this eliminating irrationality in numerator and denominator by multiplying the given fraction of the correspondent conjugate factor. Doing this very often use the formulas:

$$a^2-b^2=(a-b)(a+b),\ \ a^3\pm b^3=(a\pm b)(a^2\mp ab+b^2).$$

Example. $\lim\limits_{x\to 2}\dfrac{\sqrt{x+2}-2}{x^2-4}$.

Solution. As $x\to 2$ then $x-2\to 0$. Let's pick the factor $(x-2)$ in numerator and denominator. Let's multiply numerator and denominator of the fraction of the factor $\sqrt{x+2x}+2$. Thus in numerator we obtain the following:

$(\sqrt{x+2}-2)(\sqrt{x+2}+2)=(\sqrt{x+2})^2-2^2=x+2-4=x-2$.

In denominator the factor $\sqrt{x+2}+2$ will aim to the finite limit not equal to 0 and namely to 4 at $x\to 2$, therefore according to the theorem concerning the limit of a product the factor $\dfrac{1}{4}$ can be picked outside the sign of the limit.

Let's present denominator as the following product: $x^2-4=(x-2)(x+2)$. Thus calculation of the given limit is as follows:

$$\lim_{x\to 2}\frac{x+2-4}{\left(\sqrt{x+2}+2\right)(x-2)(x+2)}=\frac{1}{4}\lim_{x\to 2}\frac{(x-2)}{(x-2)(x+2)}=\frac{1}{4}\lim_{x\to 2}\frac{1}{(x+2)}=\frac{1}{16}.$$

Example. $\lim\limits_{x\to -2}\dfrac{x^2+2x}{\sqrt[3]{6-x}-\sqrt[3]{10+x}}=\left\|\dfrac{0}{0}\right\|$.

Solution. Let's pick out the factor aiming to 0, i.e. $x+2$.
Numerator: $x^2+2x=x(x+2)$.
Denominator: $\sqrt[3]{6-x}-\sqrt[3]{10+x}=\left\| a-b=\dfrac{a^3-b^3}{a^2+ab+b^2} \right\|=$

$$=\frac{6-x-10-x}{\sqrt[3]{(6-x)^2}+\sqrt[3]{(6-x)(10+x)}+\sqrt[3]{(6+x)^2}}=\frac{-2x-4}{\sqrt[3]{(6-x)^2}+\sqrt[3]{(6-x)(10+x)}+\sqrt[3]{(10+x)^2}}$$

.
Thus the limit is as follows

$$\lim_{x\to-2}\frac{x(x+2)\left(\sqrt[3]{(6-x)^2}+\sqrt[3]{60-4x-x^2}+\sqrt[3]{(10+x)^2}\right)}{-2\left(x+2\right)}=-\frac{1}{2}(-2)\left(\sqrt[3]{64}+\sqrt[3]{64}+\sqrt[3]{64}\right)=12$$

.
While deploying equivocations like $(\infty-\infty)$ we are to carry out identical conversions allowing to transform this uncertainty to $\frac{0}{0}$ or $\frac{\infty}{\infty}$. For example, in the case of low-level index of root we can do this by multiplying and dividing the given expression on the "conjugate" expression.

Example.

$$\lim_{x\to+\infty}\left(\sqrt{x^2+4x+1}-\sqrt{x^2+2x+1}\right)=\left\|\infty-\infty\right\|=$$

$$=\lim_{x\to+\infty}\frac{\left(\sqrt{x^2+4x+1}-\sqrt{x^2+2x+1}\right)\left(\sqrt{x^2+4x+1}+\sqrt{x^2+2x+1}\right)}{\sqrt{x^2+4x+1}+\sqrt{x^2+2x+1}}=$$

$$=\lim_{x\to+\infty}\frac{x^2+4x+1-x^2-2x-1}{\sqrt{x^2+4x+1}+\sqrt{x^2+2x+1}}=\lim_{x\to+\infty}\frac{2\frac{x}{x}}{\sqrt{\frac{x^2}{x^2}+4\frac{x}{x^2}+\frac{1}{x^2}}+\sqrt{\frac{x^2}{x^2}+2\frac{x}{x^2}+\frac{1}{x^2}}}=\frac{2}{\sqrt{1}+\sqrt{1}}=1.$$

Calculus of the limits of functions using the I-st and the II-nd remarkable limit, and also consequences from them

There are examples of the limits for which calculus is convenient to use the tables of equivalent infinitesimal obtained as consequences of the I-st and II-nd remarkable limits or directly of these limits.

Example.

$$\lim_{x\to0}\frac{tg3x}{\sqrt{1-\cos 4x}}=\left\|\frac{0}{0}\right\|=\left\|\begin{matrix}tg3x\sim 3x\\ 1-\cos 4x=2\sin^2 2x\sim 2\cdot 2x\cdot 2x\end{matrix}\right\|=\lim_{x\to0}\frac{3x}{\sqrt{2\left(2x\right)^2}}=\frac{3\sqrt{2}}{4}.$$

Example. $\lim\limits_{x\to 0}\dfrac{1-\cos^2 2x}{x\sin 3x}$.

Solution. $1-\cos^2 2x=\sin^2 2x\sim(2x)^2$, $x\sin 3x\sim x\cdot 3x$, $x\to 0$.

Then $\lim\limits_{x\to 0}\dfrac{2x\cdot 2x}{x\cdot 3x}=\dfrac{4}{3}$.

Example. $\lim\limits_{x\to 0}\dfrac{3^x-1}{arcsin2x}=\left\|\dfrac{0}{0}\right\|=\left\|\begin{matrix}3^x-1\underset{x\to 0}{\sim}x\ln 3\\ arcsin2x\underset{x\to 0}{\sim}2x\end{matrix}\right\|=\lim\limits_{x\to 0}\dfrac{x\ln 3}{2x}=\dfrac{\ln 3}{2}$.

Example. $\lim\limits_{x\to\infty}\left(\dfrac{x+4}{x-2}\right)^{\frac{5x}{2}}=\left\|1^\infty\right\|$.

Solution. While deploying equivocations like (1^∞) we should use the second remarkable limit $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x=e$.

Analysis of the expression standing under the sign of the II-nd remarkable limit shows that its structure is the following: to 1 is added the infinitesimal $\dfrac{1}{x}=\alpha$ and the obtained sum is raised in the power equal to inverse value of the added infinitesimal, i.e. x or $\dfrac{1}{\alpha}$. If in the given example we add and deduct 1 from the basis of the power then the expression will remain unchanged but we will be able to define kind of the added infinitesimal.

$$\frac{x+4}{x-2}=1+\frac{x+4}{x-2}-1=1+\frac{x+4-x+2}{x-2}=1+\frac{6}{x-2}.$$

Thus we accept as infinitesimal α the addend $\alpha=\dfrac{6}{x-2}$ aiming to 0 at $x\to\infty$. The inverse values $\dfrac{1}{\alpha}$ make $\dfrac{x-2}{6}$. Let's carry out the identical conversion:

$$\left(\frac{x+4}{x-2}\right)^{\frac{5x}{2}}=\left(\left(1+\frac{6}{x-2}\right)^{\frac{(x-2)}{6}}\right)^{\frac{6}{x-2}\cdot\frac{5x}{2}}.$$

Limit of the basis in the obtained expression makes e, i.e.

$\lim_{x\to\infty}\left(1+\frac{6}{x-2}\right)^{\frac{(x-2)}{6}}=e$ (according to the II-nd remarkable limit).

Then the given is converted to calculus the limit of the exponent, i.e.

$$\lim_{x\to\infty}\left(\left(1+\frac{6}{x-2}\right)^{\frac{(x-2)}{6}}\right)^{\frac{6\cdot 5x}{(x-2)\cdot 2}}=e^{\lim_{x\to\infty}\frac{30}{2}\cdot\frac{x}{x-2}}=e^{15}.$$

Limit of the exponent is calculated as follows:

$$\lim_{x\to\infty}\frac{30}{2}\cdot\frac{x}{x-2}=\frac{30}{2}\lim_{x\to\infty}\frac{x}{x-2}=\frac{30}{2}\lim_{x\to\infty}\frac{1}{1-\frac{2}{x}}=\frac{30}{2}=15.$$

As the conclusion of the conducted analysis we should mention that if α is infinitesimal and y is indefinitely large ($x\to a$ or $x\to\infty$), then $\lim(1+\alpha)^y=e^{\lim\alpha\cdot y}$.

Example. $\lim_{x\to\infty}\left(1+\frac{5}{3x}\right)^{2x}=\lim_{x\to\infty}\left(1+\frac{1}{\frac{3x}{5}}\right)^{2x}=\lim_{x\to\infty}\left(1+\frac{1}{\frac{3x}{5}}\right)^{\frac{3x}{5}\cdot\frac{5\cdot 2x}{3x}}=e^{\frac{10}{3}}$.

Below are considered the limits when $x\to a$, i.e. $x-a\to 0$. In this case it is convenient to change variable as follows: $x-a=z,\ z\to 0$.

Example. $\lim_{\substack{x\to 2\\(x-2)\to 0}}\frac{2^x-4}{\sin\pi x}=\left\|\frac{0}{0}\right\|=\left\|\begin{matrix}x-2=z,\ z\to 0\Rightarrow x=2+z\\ 2^x-4=2^{2+z}-2^2=2^2(2^z-1)\underset{z\to 0}{\sim}2^2z\ln 2\\ \sin\pi x=\sin(2\pi+\pi z)=\sin\pi z\underset{z\to 0}{\sim}\pi z\end{matrix}\right\|=$

$=\lim_{z\to 0}\frac{4\ln 2\cdot z}{\pi z}=\frac{4\ln 2}{\pi}$.

Example.

$$\lim_{x\to 0}\frac{\ln\cos 2x}{\ln\left(1+x^2\right)^3}=\left\|\frac{0}{0}\right\|=\left\|\begin{matrix}\cos 2x=1-1+\cos 2x=1-(1-\cos 2x)=1-2\sin^2 x\\ \ln(\cos 2x)=\ln\left(1+\left(-2\sin^2 x\right)\right)\sim -2\sin^2 x\sim -2x^2\\ \ln\left(1+x^2\right)^3=3\ln\left(1+x^2\right)\sim 3x^2, x\to 0\end{matrix}\right\|=$$

$=\lim_{x\to 0}\frac{-2x^2}{3x^2}=-\frac{2}{3}.$

Example.

$$\lim_{x\to 1}\frac{\sqrt[4]{x}-1}{x-1}=\left\|\frac{0}{0}\right\|=\left\|\begin{array}{l} x=z^4 \\ z^4-1=\left(z^2-1\right)\left(z^2+1\right)= \\ =\left(z-1\right)\left(z+1\right)\left(z^2+1\right) \\ \sqrt[4]{x}-1=z-1 \end{array}\right\|=\lim_{z\to 1}\frac{\left(z-1\right)}{\left(z-1\right)\left(z+1\right)\left(z^2+1\right)}=\frac{1}{4}.$$

Home task

Calculate the following limits of the functions:

1. $\lim\limits_{x\to\infty}\dfrac{2x^2-5x+8}{-3x^2+2x-1}$

2. $\lim\limits_{x\to 2}\dfrac{x^2-4}{x^2-2x}$

3. $\lim\limits_{x\to 0}\dfrac{\sqrt{x+9}-3}{x}$

4. $\lim\limits_{x\to 0}\dfrac{\sin 3x}{x}$

5. $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}$

6. $\lim\limits_{x\to\infty}\left(\sqrt{x^2+8x+3}-\sqrt{x^2+4x+3}\right)$

7. $\lim\limits_{x\to 3}\left(\dfrac{1}{x-3}-\dfrac{6}{x^2-9}\right)$

8. $\lim\limits_{x\to 0}\dfrac{\ln\left(1-2x\right)}{x}$

9. $\lim\limits_{x\to\infty}\left(\dfrac{x+2}{x-1}\right)^{2x}$

10. $\lim\limits_{x\to 0}\dfrac{\ln\left(x+3\right)-\ln 3}{x}$

3. THE FUNDAMENTALS OF THE DIFFERENTIAL CALCULUS FOR FUNCTIONS OF ONE VARIABLE

3.1. Derivative

A *derivative* of the function $y=f(x)$ in a point x is called the limit of the ratio of the function increment to the argument increment, while the last one approaches to zero

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}.$$

The concept of a derivative is widely applied in many areas. For example, if the function $y = f(x)$ describes the law of motion of a mass point, the derivative determines *instantaneous velocity* of the mass point in an instant x.

The elementary rules of calculus derivatives.

Let the functions $u = u(x)$ and $v = v(x)$ have their derivatives u', v' at some definite point.

1) $(cu)' = cu'$;

2) $(u \pm v)' = u' \pm v'$;

3) $(u \cdot v)' = u'v + uv'$;

4) $\left(\dfrac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}, \quad v \neq 0$;

5) $y = f(u), \; u = u(x), \quad y'_x = f'_u \cdot u'_x$;

6) $y'_x = \dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}} = \dfrac{1}{x'_y}$.

The table of derivatives of the simplest elementary functions considering the argument u being function of x is shown below:

1) $c' = 0$

2) $(u^n)' = nu^{n-1} \cdot u'$

3) $\left(\sqrt{u}\right)' = \dfrac{u'}{2\sqrt{u}}$

4) $\left(\dfrac{1}{u}\right)' = -\dfrac{u'}{u^2}$

5) $\left(\log_a u\right)' = \dfrac{u'}{u \ln a}$

6) $\left(\ln u\right)' = \dfrac{u'}{u}$

7) $\left(a^u\right)' = a^u \ln a \cdot u'$

8) $\left(e^u\right)' = e^u u'$

9) $\left(\sin u\right)' = \cos u \cdot u'$

10) $\left(\cos u\right)' = -\sin u \cdot u'$

11) $\left(\mathrm{tg} u\right)' = \dfrac{1}{\cos^2 u} \cdot u'$

12) $\left(\mathrm{ctg} u\right)' = -\dfrac{1}{\sin^2 u} \cdot u'$

13) $\left(\arcsin u\right)' = \dfrac{u'}{\sqrt{1-u^2}}$

14) $\left(\arccos u\right)' = -\dfrac{u'}{\sqrt{1-u^2}}$

15) $\left(\mathrm{arctg} u\right)' = \dfrac{u'}{1+u^2}$

16) $\left(\mathrm{arcctg} u\right)' = -\dfrac{u'}{1+u^2}$

Examples. Find derivatives of the following functions:

1) $y = \ln \sin x$.

Solution. As $\left(\ln u\right)' = \frac{1}{u}\cdot u'$, $u' = \left(\sin x\right)' = \cos x$, then $y' = \frac{1}{\sin x}\cdot \cos x$.

2) $y = \mathrm{tg}^{12}\sqrt[3]{\cos x + 4x}$.

Solution. The given function is a power function, the basis of which is a composite function, therefore the calculus of its derivative we should do sequentially using the rules of differentiating a composite function.

$y' = 12\mathrm{tg}^{11}\sqrt[3]{\cos x + 4x}\cdot \frac{1}{\cos^2\sqrt[3]{\cos x + 4x}}\cdot \frac{1}{3}(\cos x + 4x)^{-2/3}(-\sin x + 4)$.

Derivation of implicit functions

$$F(x,y) = 0.$$

In order to find derivative y' of the function $y = f(x)$ given implicitly it is necessary to differentiate both parts of the identity $F(x, y(x)) = 0$ on variable x using the rule of differentiating a composite function. Then the obtained equation should be solved by y'.

Example. Find the derivative of the function given by the following equation:

$$x\cdot y + \mathrm{tg}y = 0.$$

Solution. Differentiating on x we obtain the following:

$$y + xy' + \frac{y'}{\cos^2 y} = 0, \qquad y'\left(x + \frac{1}{\cos^2 y}\right) = -y, \qquad y' = \frac{-y\cos^2 y}{x\cos^2 y + 1}.$$

Logarithmic differentiation

Let the function $y = f(x)$ have its derivative $y' = f'(x)$ rather difficult to calculate using previously described methods and formulas, but its Napierian logarithm $\ln f(x)$ is the function that can be easily differentiated. Then, in order to find the derivative, we should use the method of logarithmic differentiating, including sequential taking the logarithm of the initial function $\ln y = \ln f(x)$ and then its differentiating as an implicit function.

Thus, if $\ln y = \varphi(x)$, then $\frac{y'}{y} = \varphi'(x)$

whence we find $y' = y\cdot\varphi'(x)$ or $y' = f(x)\cdot\varphi'(x)$.

Example. Find the derivative of the following function: $y = \left(x^3 + 5x^2\right)^{\frac{1}{x}}$.

Solution. The formula for differentiating the given function isn't present in the table. Therefore we use the method of logarithmic differentiating. Let's take the logarithm of this function:

$$\ln y = \frac{1}{x}\ln\left(x^3 + 5x^2\right).$$

Differentiating both parts of the equation we obtain the following:

$$\frac{y'}{y} = -\frac{1}{x^2}\ln\left(x^3 + 5x^2\right) + \frac{1}{x}\cdot\frac{3x^2 + 10x}{x^3 + 5x^2},$$

whence

$$y' = \left(x^3 + 5x^2\right)^{\frac{1}{x}}\left(-\frac{1}{x^2}\ln\left(x^3 + 5x^2\right) + \frac{3x \;\; + 10}{x^3 + 5x^2}\right).$$

Example. $y = \sqrt[3]{\dfrac{x(x^2 + 1)}{(x^2 - 1)^2}}$.

Solution. The direct calculus of the derivative of the given function is rather difficult while the Napierian logarithm y can by easily differentiated. Let's take the logarithm of this function:

$$\ln y = \frac{1}{3}\left(\ln x + \ln(x^2 + 1) - 2\ln(x^2 - 1)\right).$$

Considering y as the function of x we differentiate both parts of the following identity:

$$\frac{y'}{y} = \frac{1}{3}\left(\frac{1}{x} + \frac{2x}{x^2 + 1} - 2\frac{2x}{x^2 - 1}\right),$$

whence

$$y' = \sqrt[3]{\frac{x(x^2 + 1)}{(x^2 - 1)^2}}\cdot\frac{1}{3}\left(\frac{1}{x} + \frac{2x}{x^2 + 1} - 2\frac{2x}{x^2 - 1}\right).$$

The geometrical meaning of a derivative. The equations of the tangent and normal line

Derivative of a function at the given point is numerically equal to the angular coefficient of the tangent to the curve at this point. Therefore *the equation of non-vertical tangent line* to the curve $y = f(x)$ at the point x_0 is as follows:

$$y - y_0 = y'(x_0)(x - x_0).$$

The equation of a vertical tangent line is $x = x_0$.

The normal line to the curve at the point $M_0(x_0, y_0)$ is a perpendicular to the tangent of this curve at the given point.

The equation of a non-horizontal normal line looks like $y - y_0 = -\frac{1}{y'(x_0)}(x - x_0)$.

The equation of a horizontal normal is $y = y_0$.

Example. Find the equations of a tangent and normal lines to the curve $y = x^3 - 3x^2 - 2$ at the point with the abscissa $x_0 = 1$.

Solution. The ordinate of the tangency point is $y_0 = 1^3 - 3 \cdot 1^2 - 2 = -4$. Angular coefficient of the tangent line makes

$$k = y'\Big|_{x=1} = \left(3x^2 - 6x\right)\Big|_{x=1} = 3 - 6 = -3 \ .$$

The equation of the tangent is as follows:

$$y + 4 = -3(x - 1), \quad \text{or} \quad 3x + y + 1 = 0 .$$

Angular coefficient of the normal line $k_{norm} = -\frac{1}{k_{\tan g}} = \frac{1}{3}$. The equation of the normal line makes $y + 4 = \frac{1}{3}(x - 1)$, or $x - 3y - 13 = 0$.

3.2. Differential of a function

The function $y = f(x)$ is considered to be differential at the given point x if the increment Δy of this function at the point x corresponding to the increment of the argument Δx can be shown as:

$$\Delta y = A\Delta x + \alpha \Delta x ,$$

where A is some value, not depending upon Δx, and α is a function of the argument Δx being infinitesimal as Δx aims to zero. The main part of the function increment $A \cdot \Delta x$, linear relatively Δx, is called the *differential of the function* and is designated as $dy = A\Delta x$.

The differential of the variable x is equal to its increment $dx = \Delta x$, therefore

$$dy = y'dx .$$

As we see $y' = \frac{dy}{dx}$, i.e. the derivative of the function may be regarded as the ratio of the function differential y to the differential (increment) of the independent variable dx.

Example. Find a differential of the following function: $y = \ln \mathrm{tg}\sqrt{x}$.

Solution. $dy = \dfrac{dx}{\mathrm{tg}\sqrt{x} \cdot \cos^2\sqrt{x} \cdot 2\sqrt{x}} = \dfrac{dx}{\sqrt{x} \cdot \sin 2\sqrt{x}}$.

Differentiating of the functions given in parametric form

If the function is given in parametric form, i.e. $\begin{cases} x = x(t), \\ y = y(t), \end{cases}$

then its derivative on x can be presented in the following way:

$$y'_x = \frac{dy}{dx} = \frac{y'_t dt}{x'_t dt} = \frac{y'_t}{x'_t}, \text{ i.e. } y'_x = \frac{y'_t}{x'_t}. \quad y''_{xx} = \frac{y''_{tt}\, x'_t - x''_{tt}\, y'_t}{(x'_t)^3}.$$

Example. Find the derivative y'_x, if the function is given in parametric form $\begin{cases} x = a(t - \sin t), \\ y = a(1 - \cos t). \end{cases}$

Solution. Then $x'_t = a(1-\cos t)$, $y'_t = a\sin t$, and $y'_x = \dfrac{a\sin t}{a(1-\cos t)} = \dfrac{\sin t}{1-\cos t}$.

Geometrical meaning of a differential of a function

The differential of the function $y = f(x)$ makes $dy = f'(x)dx$. Taking into consideration, that $f'(x) = \mathrm{tg}\varphi$ (Fig.5) we obtain $dy = \mathrm{tg}\varphi dx$, i.e. geometrical meaning of a differential as its equality to the increment in the ordinate of the line tangent to the curve $y = f(x)$ at the point with the abscissa x.

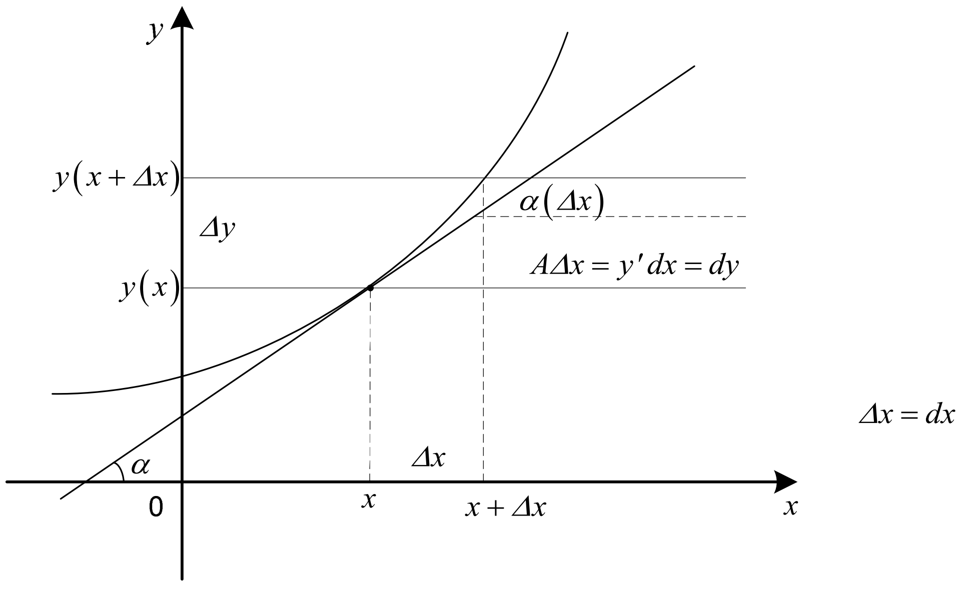


Fig.5

The application of the differential to approximate calculations

If Δx is small enough it is possible to use the differential of the function in the state of its increment, i.e. $f(x_0+\Delta x)-f(x_0)\approx f'(x_0)\Delta x$ and then get an approximate value of the required according to the next formula:

$$f(x_0+\Delta x)\approx f(x_0)+f'(x_0)\Delta x.$$

Example. Calculate an approximate value of arctg 0,97.

Solution. $\text{arctg}(x_0+\Delta x)\approx \text{arctg}\,x_0+\text{arctg}'(x_0)\Delta x$;

$x_0+\Delta x=0{,}97$; $x_0=1$; $\Delta x=-0{,}03$: $(\text{arctg}x)'=\dfrac{1}{1+x^2}$.

$$\text{arctg}0{,}97\approx \text{arctg}1-\frac{0{,}03}{1+1^2}=\frac{\pi}{4}-0{,}015\approx 0{,}7554.$$

3.3. Derivatives and differentials of the higher orders

Let the function $y=f(x)$ be differentiable in some interval (a,b). Generally, the value of the derivative $f'(x)$ depends on x, i.e. the derivative $f'(x)$ is also a function of x. If this function is differentiable at some point x of the interval (a,b), i.e. has the derivative at this point, then this derivative is called the second derivative (or second order derivative) and is designated as:

$$y''=(y')'=f''(x).$$

The same way we can introduce the concept of the third order derivative, then the concept of the fourth order derivative, etc.

The 2$^{\text{nd}}$ order differential is called the differential of the differential of a function, i.e.

$$d(dy)=d(y'dx)=y''dx^2=d^2y$$

or

$$d^2y=y''dx^2.$$

Generally, the n^{th} order differential is called the first differential of a differential of $(n-1)^{\text{th}}$ order.

$$d^ny=d(d^{n-1}y)=y^{(n)}dx^n.$$

3.4. Disclosing indeterminate forms by L'Hospital's rule

The rule by L'Hospital for disclosing indeterminate forms $\left(\frac{0}{0}\right)$ and $\left(\frac{\infty}{\infty}\right)$ is formulated as the following theorem.

The theorem. Let the single-valued functions $f(x)$ and $\varphi(x)$ be differentiable everywhere in some neighborhood of the point a, i.e. at $0<|x-a|<\varepsilon$ and $\varphi'(x)\neq 0$, then if there exists the limit (finite or infinite) of the ratio of the derivatives, the ratio of the functions has the same limit.

$$1)\ \lim_{x\to a}\frac{f(x)}{\varphi(x)}=\left(\left\|\frac{0}{0}\right\| \text{ or } \left\|\frac{\infty}{\infty}\right\|\right)=\lim_{x\to a}\frac{f'(x)}{\varphi'(x)};$$

$$2)\ \lim_{x\to \infty}\frac{f(x)}{\varphi(x)}=\left(\left\|\frac{0}{0}\right\| \text{ or } \left\|\frac{\infty}{\infty}\right\|\right)=\lim_{x\to \infty}\frac{f'(x)}{\varphi'(x)}.$$

Note. Let's note once again, that the existence of a limit of the ratio of the derivatives guarantees the existence of the functions ratio limit. The converse is false, i.e. *the limit of the ratio of functions can exist while non-existence of the derivatives ratio limit.*

Example. $$\lim_{x\to\infty}\frac{\pi-2arctgx}{e^{\frac{2}{x}}-1}=\left\|\frac{0}{0}\right\|=\lim_{x\to\infty}\frac{-2\frac{1}{1+x^2}}{e^{\frac{2}{x}}\cdot\left(-\frac{2}{x^2}\right)}==\frac{2}{2}\lim_{x\to\infty}\frac{x^2}{1+x^2}=1.$$

The rule by L'Hospital can be applied sequentially several times if the ratio of the derivatives again leads to indeterminate and derivatives satisfy the conditions of the rule by L'Hospital.

Example. $$\lim_{x\to 0}\frac{x-tgx}{2x^3}=\left\|\frac{0}{0}\right\|=\lim_{x\to 0}\frac{1-\frac{1}{\cos^2 x}}{6x^2}=\left\|\frac{0}{0}\right\|=\lim_{x\to 0}\frac{-2\frac{\sin x}{\cos^3 x}}{12x}=-\frac{1}{6}.$$

Example.

$$\lim_{x\to a}\frac{\ln(x-a)}{\ln(e^x-e^a)}=\left\|\frac{\infty}{\infty}\right\|=\lim_{x\to a}\frac{\frac{1}{x-a}}{\frac{1}{e^x-e^a}e^x}=e^{-a}\cdot\lim_{x\to a}\frac{e^x-e^a}{x-a}=\left\|\frac{0}{0}\right\|=e^{-a}\cdot\lim_{x\to a}\frac{e^x}{1}=1.$$

Disclosing of the indeterminate forms $\underline{0\cdot\infty}$ and $\underline{\infty-\infty}$ is realized with the help of identical transformations that convert these indeterminate forms to $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and then apply the table of equivalent infinitesimal values and the rule by L'Hospital.

a) $$\lim_{x\to x_0} f(x)\varphi(x)=\|0\cdot\infty\|=\lim_{x\to x_0}\frac{f(x)}{\frac{1}{\varphi(x)}}=\left\|\frac{0}{0}\right\|=\lim_{x\to x_0}\frac{f'(x)}{-\frac{\varphi'(x)}{\varphi^2(x)}};$$

b) $$\lim_{x\to x_0} f(x)\varphi(x)=\|0\cdot\infty\|=\lim_{x\to x_0}\frac{\varphi(x)}{\frac{1}{f(x)}}=\left\|\frac{\infty}{\infty}\right\|=\lim_{x\to x_0}\frac{\varphi'(x)}{-\frac{f'(x)}{f^2(x)}}.$$

Example $\|\infty-\infty\|$**.**

$$\lim_{x\to1}\left(\frac{1}{x-1}-\frac{1}{\ln x}\right)=\lim_{x\to1}\frac{\ln x-x+1}{(x-1)\ln x}=\left\|\frac{0}{0}\right\|=\lim_{x\to1}\frac{\frac{1}{x}-1}{\ln x+(x-1)\frac{1}{x}}=\lim_{x\to1}\frac{\frac{-1}{x^2}}{\frac{1}{x}+\frac{1}{x^2}}=-\frac{1}{2}.$$

Disclosing the indeterminate forms ∞^0; 0^0; 1^∞ is realized with previous transformation of power – exponential expression with the help of the basic logarithmic identity $a\equiv e^{\ln a}$. As a result of these operations we obtain the following:

a) $\infty^0=e^{\ln\infty^0}=e^{0\cdot\ln\infty}=e^{0\cdot\infty}=e^{\frac{0}{1/\infty}}=e^{\frac{0}{0}}$;

b) $0^0=e^{\ln 0^0}=e^{0\cdot\ln 0}=e^{0\cdot\infty}=e^{\frac{0}{1/\infty}}=e^{\frac{0}{0}}$;

c) $1^\infty=e^{\ln 1^\infty}=e^{\infty\cdot\ln 1}=e^{\infty\cdot 0}=e^{\frac{0}{1/\infty}}=e^{\frac{0}{0}}$.

While calculating the limits usually combine the *application of equivalent infinitesimal* and the *rule by L'Hospital*. All factors, seeking to the finite nontrivial limits, are *immediately substituted* by these limits.

Example**.**

$$\lim_{x\to0}x^{\sin x}=\|0^0\|=e^{\lim\limits_{x\to0}\sin x\ln x}=e^{\lim\limits_{x\to0}\frac{\ln x}{\frac{1}{\sin x}}}=e^{\lim\limits_{x\to0}\frac{\frac{1}{x}}{\frac{-\cos x}{\sin^2 x}}}=e^0=1.$$

Example**.** $$\lim_{x\to0}(\cos 2x)^{\frac{3}{x^2}}=\|1^\infty\|=e^{\lim\limits_{x\to0}\frac{3}{x^2}\ln\cos 2x}=\left\|e^{\frac{0}{0}}\right\|=e^{3\lim\limits_{x\to0}\frac{\frac{1}{\cos 2x}(-\sin 2x)\cdot 2}{2x}}=$$

$$=\left\|\begin{matrix}\sin 2x \sim 2x \\ cos 2x \underset{x\to 0}{\to} 1\end{matrix}\right\| = e^{-3\lim\limits_{x\to 0}\frac{2x}{x}} = e^{-6}.$$

3.5. The functions investigation by means of derivatives

Requirement of the monotonicity of the function. The extremums.

1. Let the function $f(x)$ be continuous and differentiable in the interval $[a,b]$. In order to the function to be a constant in $[a,b]$ it is necessary and enough that $f'(x)=0 \ \forall x \in (a,b)$.

2. Let the function $f(x)$ be continuous and differentiable in the interval $[a,b]$, then a) if $f'(x)>0 \ \forall x \in (a,b)$, then $f(x)$ increases; b) if $f'(x)<0 \ \forall x \in (a,b)$, then $f(x)$ decreases.

3. Let the function $f(x)$ is differentiable and increases in the interval $[a,b]$, then $f'(x) \geq 0 \ \forall x \in (a,b)$. If the function $f(x)$ decreases then $f'(x) \leq 0 \ \forall x \in (a,b)$.

The point x_0 is called *the point of the local maximum (minimum)* of the function $y = f(x)$, if there is such in its neighborhood ($x_0 - \delta, x_0 + \delta$), in which $f(x_0)$ is its maximum (minimum) value among all other values of this function. The points of the local maximum and minimum of a function are called the points of *extremum* of this function

4. (*Necessary existence condition of an extremum*)

It the continuous function $f(x)$ has an extremum at some point, then the derivative of the function at this point either is equal to zero or does not exist. *Points*, in which the derivative is equal to zero or does not exist, are called *critical.*

5. (*Sufficient existence condition of an extremum of the function by the first derivative*)

Let x_0 be a critical point. Then, if the function $f(x)$ has its derivative $f'(x)$ in some neighborhood of the point x_0 and if the derivative $f'(x)$ while passing through the point $x = x_0$ changes its sign from a plus to a minus, the function has the maximum at this point, and at change of the sign from a minus to a plus has the minimum.

6. (*Sufficient existence condition of an extremum of the function by the second derivative*)

If the function $f(x)$ in some neighborhood of the point x_0 is continuous, has the second derivative and $f'(x_0)$=0, $f''(x_0) \neq 0$, then , if $f''(x_0) > 0$, the function has the minimum at the point x_0 and if $f''(x_0) < 0$, the function has the maximum at the point x_0.

Convexity and concavity of a curve. Point of inflection

The curve is called convex at the point x_0 if in some neighborhood of this point ($x_0 - \delta, x_0 + \delta$) it is located below the tangent (Fig.6), drawn at the point x_0. If the curve is located above the tangent, it is called concave.

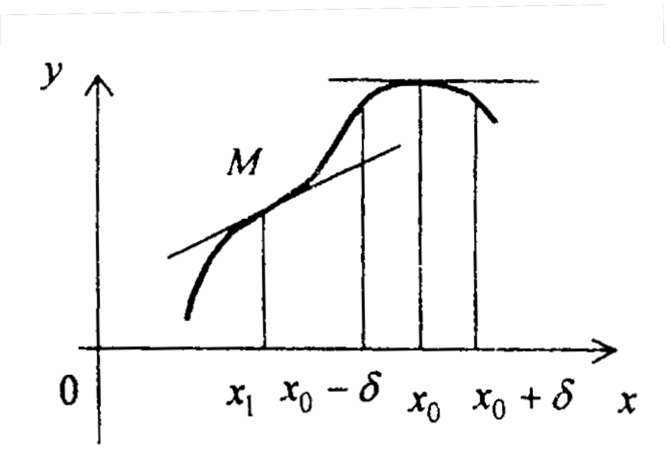


Fig.6

1. If the function $f(x)$ in some neighborhood of the point x_0 is doubly continuous differentiable and $f''(x) \neq 0$, then the necessary and sufficient condition of convexity of the curve at the point x_0 is the requirement $f''(x_0) < 0$, concavity – $f''(x_0) > 0$.

The point $M(x_1, f(x_1))$ is called *the point of inflection* of the given curve (Fig.2) if there is such neighborhood of the point x_1, that while $x < x_1$ in this neighborhood the concavity of the curve is directed to one direction, and while $x > x_1$ is directed to another one (Fig.6).

In order to let the point $x = x_0$ be the point of inflection of the given curve it is necessary that the second derivative in this point will be either equal to zero $f''(x_0) = 0$ or don't exist.

2. (Sufficient existence condition of *a point of inflection*)

Let the curve be defined by the equation $y = f(x)$. If $f''(x_0) = 0$ or $f''(x_0)$ does not exist and while passing through $x = x_0$ the derivative $f''(x)$ changes its sign, the point of the curve with the abscissa x_0 is the point of inflection.

Asymptotes of curves

The straight line $x = x_0$ is called a *vertical asymptote* if $\lim\limits_{x \to x_0} f(x) = \pm\infty$.

Example. $y = \dfrac{1}{1 - x^2}$.

Solution. The straight lines $x = \pm 1$ are vertical asymptotes, i.e. $\lim\limits_{x \to 1 \pm 0} \dfrac{1}{1 - x^2} = \mp\infty$; $\lim\limits_{x \to -1 \pm 0} \dfrac{1}{1 - x^2} = \pm\infty$.

The asymptote of the plot of the function $y = f(x)$ is a straight line, possessing such property that the distance between the line and the point on the curve aims to zero while the point on the curve unrestrictedly leaves in perpetuity ($x \to \pm\infty$).

The equation of the *inclined asymptote* looks like $y = kx + b$. In particular, if $k = 0$, the asymptote is horizontal. If the inclined asymptote exists, k and b are to be calculated according to the following formulas:

$$k = \lim_{x \to \pm\infty} \frac{f(x)}{x}, \quad b = \lim_{x \to \pm\infty} (f(x) - kx).$$

If at least one limit doesn't exist, the curve has no inclined asymptotes. The asymptotes can vary while $x \to +\infty$ and $x \to -\infty$.

General plan for investigating a function and constructing its plot

1) Defining the existence area of the function. $D\{y\}$, $E\{y\}$, x=0, y=0. Investigating the function on the continuity.

2) Investigating the function on parity, oddness and periodicity.

3) Investigating the function on extremums. Finding the intervals of the monotonicity of the function. (y'), ↗, ↘.

4) Finding the inflection points of the function, intervals of convexity and concavity. (y''), $\cap$, $\cup$.

5) Defining the vertical (|), horizontal (–) and inclined (/) asymptotes.

Example. Plot the graph of the function: $y = \dfrac{x^3}{2\left(x-1\right)^2}$.

Solution.

1. $D\{y\} : (x-1)^2 \neq 0, \; x \neq 1.$ $\qquad E\{y\} : y \in \left(-\infty, \infty\right)$

$x = 1$ is a point of discontinuity of the function $\lim\limits_{x \to -1 \pm 0} \dfrac{x^3}{2(x-1)^2} = \infty$, and therefore $x = 1$ is a vertical asymptote. $\qquad x = 0; \; y = 0$.

2. $y(-x) = \dfrac{(-x)^3}{2(-x-1)^2} = -\dfrac{x^3}{2(x+1)^2}; \quad y(-x) \neq y(x), \;\; y(-x) \neq -y(x)$. It's function of the general view.

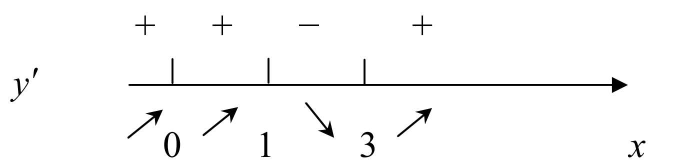
Function is acyclic, i.e. there is no such value T that the equality $f(x+T) = f(x)$, $\forall x \in D(f)$ takes place.

3. For defining the intervals of monotonicity and extremums of the function it is necessary to find the points at which its first derivative is equal to zero or does not exist:

$$y' = \frac{3x^2(x-1)^2 - 2(x-1)x^3}{2(x-1)^4} = \frac{3x^3 - 3x^2 - 2x^3}{2(x-1)^3} = \frac{x^2\left(x-3\right)}{2(x-1)^3},$$

$$y' = \frac{x^2(x-3)}{2(x-1)^3} = 0,$$

$x^2 = 0$, $x=0$, $x - 3=0$, $x= 3$, $x - 1 \neq 0$, $x \neq 1$.



At $x \in (-\infty;0) \cup (0;1) \cup (3;+\infty)$ the function increases; at $x \in (1;3)$ the function decreases. $\qquad y_{min}(3) = \dfrac{27}{2 \cdot 4} = \dfrac{27}{8}$.

4. For defining the intervals of convexity/concavity and points of inflection we shall find the second derivative:

$$y'' = \frac{1}{2}\frac{\left(x-1\right)^3\left(3x^2-6x\right) - \left(x^3-3x^2\right)3\left(x-1\right)^2}{(x-1)^6} = \frac{3x}{(x-1)^4} = 0, \;\; x = 0, \;\; x \neq 1.$$

$$\begin{array}{cccc} & - & + & + \\ y'' & \cap \;\; 0 & \cup \;\; 1 & \cup \quad x \end{array}$$

At $x \in (-\infty;0)$ the graph is convex; at $x \in (0;1) \cup (1,\infty)$ the graph is concave.

The point $O(0;0)$ – is a point of inflection.

The points of intercrossing of the graph with coordinate axes are: $x = 0$, $y = 0$.

5. Let's investigate the function on infinity: $\lim\limits_{x \to \pm\infty} \dfrac{x^3}{2(x-1)^2} = \pm\infty$.

Therefore, the horizontal asymptotes are apsant.

The inclined asymptotes $y = kx + b$.

$$k = \lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} \frac{x^2}{2(x-1)^2} = \lim_{x \to \pm\infty} \frac{1}{2\left(1-\frac{1}{x}\right)^2} = \frac{1}{2},$$

$$b = \lim_{x \to \pm\infty} \left(f(x) - kx\right) = \lim_{x \to \pm\infty} \left(\frac{x^3}{2(x-1)^2} - \frac{1}{2}x\right) \left|\infty - \infty\right| = \lim_{x \to \pm\infty} \frac{x^3 - x(x-1)^2}{2(x-1)^2} =$$

$$= \lim_{x \to \pm\infty} \frac{x^3 - x^3 + 2x^2 - x}{2(x-1)^2} = \lim_{x \to \pm\infty} \frac{2x^2 - x}{2(x-1)^2} = \lim_{x \to \pm\infty} \frac{2 - \frac{1}{x}}{2\left(1-\frac{1}{x}\right)^2} = 1.$$

$y = 1/2x + 1$. This is inclined asymptote. The graph of the function − Fig.7.

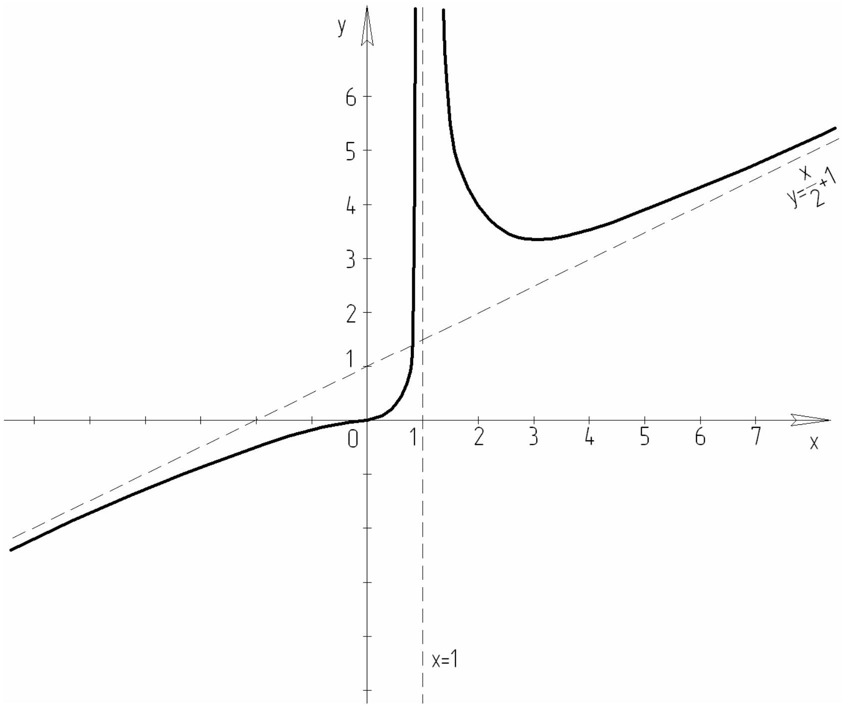


Fig.7

3.6. Minimum and maximum of a function in an interval

Let the function $y = f(x)$ be continuous in the interval $[a,b]$. Then, according to the theorem by Weierstrass, it achieves maximum and minimum values in this interval. These values can be achieved either on the borders of the interval or at internal points, being the extremums of the function. Out of this fact we conclude the following plan for defining maximum and minimum values of the function:

1. Define all critical points, belonging to the given interval *(a,b).*

2. Calculate the values of the function in the found critical points and on the borders of the interval.

3. Choose maximum and minimum from the obtained values. The chosen are the needed values.

Example. Find maximum and minimum of the function $f(x) = 2x^3 - 3x^2 + 4$ in the interval $[-1,2]$.

Solution. $f'(x) = 6x^2 - 6x = 6x(x-1)$, $6x(x-1) = 0$ at $x = 0$ and $x = 1$.
Thus the given function has two stationary points $x_1 = 0$ and $x_2 = 1$ inside the interval $[-1, 2]$. Let's calculate the function values at these points and on the borders of the interval.

$$f(0) = 4; \quad f(1) = 3; \quad f(-1) = -1; \quad f(2) = 8.$$

As we see, the function takes maximum value on the right border of the interval $[-1;2]$ and minimum value on the left border of the interval $[-1;2]$.

Many problems of geometrical and physical matter are solved with the help of the theory of extremums. Let's assume we have two values, connected by functional relation, and we are to find the value of one of them (restricted in some interval included unlimited), while another one takes maximum or minimum.

For solving such problems first of all we should create an expression, describing functional relation between the functions (one function expressed through another) and then find maximum and minimum in the given interval.

Example. What should be the sizes of cylinder, that at the given certain volume the complete surface will be minimum?

Solution. Let's sign as R the radius of the cylinder basis and as H its altitude. Then the complete surface makes as follows:

$$S = 2\pi R^2 + 2\pi RH .$$

Considering the volume given according to the condition of the task, we can find one of the needed values, for example H, that can be taken through R from the formula $V=\pi R^2 H$, whence $H=\dfrac{V}{\pi R^2}$. Substituting this expression in the formula for S we obtain: $S=2\left(\pi R^2+\dfrac{V}{R}\right)$,

i.e. S is the function of a single variable R, where $R\in(0,\infty)$.

Let's find the minimum of the function in the given interval

$\dfrac{dS}{dR}=2\left(2\pi R-\dfrac{V}{R^2}\right)$; $\quad 2\pi R-\dfrac{V}{R^2}=0$. Whence $R=\sqrt[3]{\dfrac{V}{2\pi}}$. $\dfrac{d^2S}{dR^2}=2\left(2\pi+\dfrac{2V}{R^3}\right)>0$ or $S\to\infty$ at $R\to 0$ and $S\to\infty$ at $R\to\infty$ and inside the interval we have the only stationary point, then we come to a conclusion, that at $R=\sqrt[3]{\dfrac{V}{2\pi}}$ the complete surface S will be minimum. Thus $H=\dfrac{V}{\pi R^2}=2\sqrt[3]{\dfrac{V}{2\pi}}$.

Home task

Find derivatives of the following function

1. $y=arctg\sqrt{x^2-1}$
2. $y=e^{sin^2 2x}$
3. $y=\dfrac{x^3}{1-x^2}$
4. $y=\dfrac{2\sqrt{x^2+1}}{x}$
5. $y=ln\,sin\,e^x$
6. $y=2^x\cdot cos\,2x$
7. $e^y=2x+y$
8. $xy-tg(x-y)=0$
9. $\begin{cases}x=arcsin\,t\\ y=arccos\,t\end{cases}$
10. $\begin{cases}x=\dfrac{1}{3}cos^3 t\\ y=\dfrac{1}{3}sin^3 t\end{cases}$

Calculate the following limits of the functions

1. $\lim_{x\to 2}\frac{x-2}{x^3-8}$

2. $\lim_{x\to 0} x\ln x$

3. $\lim_{x\to 0}\frac{1-\cos 2x}{1-\cos 3x}$

4. $\lim_{x\to 0}\frac{\arcsin x}{tgx}$

5. $\lim_{x\to 1}(x-1)^{x-1}$

6. $\lim_{x\to 0}\left(\ln\frac{1}{x}\right)^x$

Calculate approximate values

1. $\sqrt[3]{9}$

2. $(5{,}01)^3$

3. $\sin 31^0$

4. $arctg\,0{,}95$

5. $2^{2{,}1}$

6. $\arcsin 0{,}52$

Plot the graphs of the functions

1. $y=\frac{x^2-x+1}{x-1}$

2. $y=\frac{x^3-4x}{3x^2-4}$

3. $y=\frac{\sqrt{x^2-4}}{2-x}$

4. $y=\frac{x}{\sqrt[3]{x+1}}$

5. $y=arctgx-x$

6. $y=e^x+e^{-x}$

4. INDEFINITE INTEGRAL. METHODS OF INTEGRATION

4.1. Definition. Properties

Let $F(x)=\sin x$. Earlier we found the derivative of $F(x)$

$F'(x)=(\sin x)'=\cos x=f(x)$.

Now we are given a function $f(x)=\cos x$ and we need to know the derivative of which function it appears.

This statement should be written like:

$$\int f(x)dx=F(x)+c \text{ or } \int\cos xdx=\sin x+c.$$

The function $F(x)$ is called the antiderivative of the interval (a,b) if $F(x)$ is differentiable $\forall x \in (a,b)$ and $F'(x) = f(x)$.

The set of all antiderivatives $F(x)+c$ is called the indefinite integral and is designated by the symbol $\boxed{\int f(x)dx = F(x)+c\,.}$

Finding of the indefinite integral is called function integration. The integration operation is presented as an inverse differentiation.

The properties of an indefinite integral.

1. $\left(\int f(x)dx\right)' = f(x)$
2. $d\left(\int f(x)dx\right) = f(x)dx$
3. $\int dF(x) = F(x)+c$
4. $\int af(x)dx = a\int f(x)dx$, $a = const$
5. $\int\left[f_1(x) \pm f_2(x)\right]dx = \int f_1(x)dx \pm \int f_2(x)dx$
6. *If* $\int f(x)dx = F(x)+c$ and $u = \varphi(x)$, then

$\int f(u)du = F(u)+c$ − this is a property of invariation formulae of integral and this means that

If $\int cos\,xdx = sin\,x + c$, then

$$\int cos\,\square\; d\;\square = sin\,\square + C$$

Where $\square$ can be any continuous function of x

The table of basic integrals

I. $\int dx = x + C$

II. $\int x^m dx = \dfrac{x^{m+1}}{m+1} + C$ при $m \neq -1$

III. $\int \dfrac{dx}{x} = ln|x| + C$

IV. $\int \dfrac{dx}{1+x^2} = arctgx + C$

XIV. $\int \dfrac{dx}{ch^2 x} = thx + C$

XV. $\int \dfrac{dx}{sh^2 x} = -cthx + C$

XVI. $\int \dfrac{f'(x)}{f(x)}dx = ln|f(x)| + C$

XVII. $\int \dfrac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)} + C$

V. $\int \frac{dx}{\sqrt{1-x^2}} = arcsin\, x + C$

VI. $\int e^x dx = e^x + C$

VII. $\int a^x dx = \frac{a^x}{ln\, a} + C$

VIII. $\int sin\, x dx = -cos\, x + C$

IX. $\int cos\, x dx = sin\, x + C$

X. $\int sec^2\, x dx = tgx + C$

XI. $\int cos\, ec^2 x dx = -ctgx + C$

XII. $\int shx dx = chx + C$

XIII. $\int chx dx = shx + C$

XVIII. $\int \frac{dx}{x^2+a^2} = \frac{1}{a} arctg \frac{x}{a} + C$

XIX. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} ln\left|\frac{x-a}{x+a}\right| + C$

XX. $\int \frac{dx}{\sqrt{a^2-x^2}} = arcsin \frac{x}{a} + C$

XXI. $\int \frac{dx}{\sqrt{x^2+\lambda}} = ln\left|x + \sqrt{x^2+\lambda}\right| + C$

XXII. $\int \frac{dx}{sin\, x} = ln\left|tg \frac{x}{2}\right| + C = ln\left|cos\, ecx - ctgx\right| + C$

XXIII. $\int \frac{dx}{cos\, x} = ln\left|tg\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| = ln\left|sec\, x + tgx\right| + C$

XXIV. $\int tgx dx = -ln\left|cos\, x\right| + C$

XXV. $\int ctgx dx = ln\left|sin\, x\right| + C$

The application of basic integrals table is called direct integration.

Examples.

1. $\int\left(3\sqrt[3]{x} - \frac{4}{\sqrt[5]{x^3}} + 2x\right)dx \underset{=======}{\overset{prop.\ 4,5}{}} 3\int x^{\frac{1}{3}} dx -$

$$-4\int x^{-\frac{3}{5}} dx + 2\int x dx \underset{=======}{\overset{form.\ II}{}} \frac{3\cdot 3x^{\frac{4}{3}}}{4} - \frac{5\not{4}^2 x^{\frac{2}{5}}}{\not{2}} + \frac{\not{2}x^2}{\not{2}} + c$$

$$\int(x-2)^2 dx = \int\left(x^2 - 4x + 4\right)dx = \int x^2 dx - 4\int x dx + 4\int dx =$$

$$= \frac{x^3}{3} - 4\frac{x^2}{2} + 4x + c;$$

2. $\int(x-2)^{50} dx =$ one can follow example 2 and present $(x-2)^{50}$ as sum of 51 summand but it's better to give a change of variable:

$$= \left|\begin{matrix} x-2=t \\ dx=dt \end{matrix}\right| = \int t^{50} dt = \frac{t^{51}}{51} + c = \frac{(x-2)^{51}}{51} + c$$

$x - 2 = t$, where t is a new variable.

Taking into account that the function differential equals function derivative multiplied by the differential of independent variable. Let's show under the sign of differential we can add or subtract any constant in this case the differential won't change.

$dF(x) = F'(x)dx$

$$d\left(F(x) \pm c\right) = \left(F(x) \pm c\right)' dx = \left(F'(x) \pm \underset{0}{\underset{/\!/}{c'}} \right) dx = F'(x)dx = dF(x)$$

4. $\int (x-2)^{50} dx = \int (x-2)^{50} d(x-2) = \frac{(x-2)^{51}}{51} + c$

4.2. Integration by substitution or change of variable

1. Let $x = \varphi(t)$ − is continuously differentiable and monotonous function of new variable t.

Then

$$\int f(x)dx = \int f(\varphi(t))\varphi'(t)dt$$

Example

a) $\int \frac{\cos\sqrt[3]{x}}{\sqrt[3]{x^2}} dx = \left| \begin{array}{l} \sqrt[3]{x} = t \\ x = t^3 \\ dx = 3t^2 dt \end{array} \right| = \int \frac{\cos t \cdot 3t^2 dt}{t^2} = 3\sin t + c = 3\sin\sqrt[3]{x} + c$

b) $\int \frac{dx}{x\sqrt{x-4}} = \left| \begin{array}{l} \sqrt{x-4} = t; \\ x - 4 = t^2 \\ dx = 2tdt \end{array} \right| = \int \frac{2t\,dt}{(t^2+4)\cdot t} = 2\int \frac{dt}{t^2 + 2^2} = \frac{2}{2} arctg\frac{t}{2} + C =$

$= arctg\frac{\sqrt{x-4}}{2} + C$

2. $u = \psi(x)$, where u − new variable then $\int f\left[\psi(x)\right]\psi'(x)dx = \int f(u)du$

Example $\int \ln^4 x \frac{1}{x} dx = \left| \begin{array}{l} \ln x = u \\ \frac{1}{x} dx = du \end{array} \right| = \int u^4 du = \frac{u^5}{5} + c = \frac{\ln^5 x}{5} + c$.

Note. If $\int f(x)dx = F(x) + c$ then

$\int f(ax+b)dx = \frac{1}{a}\int f(ax+b)d(ax+b) = \frac{1}{a}F(ax+b) + c$.

Example $\int cos(2x+3)dx = \frac{1}{2}sin(2x+3)+c$.

4.3. Method of integration by parts

Let the function $u = u(x)$ and $v = v(x)$ have continuous derivatives, then

$$\boxed{\int udv = uv - \int vdu}$$

The integration by parts is applied in the cases when the newly received integral $\int vdu$ is more simplified or similar to the initial $\int udv$

Example

a) $$\int \underbrace{x}_{n}\underbrace{cos\,xdx}_{dv} = \left| \begin{array}{c} x = u \\ cos\,xdx = dv \\ dx = du \\ v = sin\,x + \underset{0}{\cancel{c}} \end{array} \right| = \underbrace{x}_{n}\underbrace{sinx}_{v} - \int \underbrace{sin\,x}_{v}\underbrace{dx}_{du} = x\,sin\,x \not- cos\,x + c;$$

b) Let's try it *v.v.*

$$(!)\int \underbrace{cos\,x}_{u_i}\underbrace{xdx}_{dv_i} = \left| \begin{array}{c} cos\,x = u_1 \\ xdx = dv_1 \\ du_1 = -sin\,xdx \\ v_1 = \frac{x^2}{2} \end{array} \right| = \underbrace{\frac{x^2}{2}}_{v_1}\cdot\underbrace{cos\,x}_{u_1} - \int \underbrace{\frac{x^2}{2}}_{v_1}\underbrace{(-sin\,x)dx}_{du_1}$$

In the result of the incorrect application, of the formula of integration by parts, the newly received integral to be more complicated then the initial one.

Therefore we can try to give some recommendation for certain cases

$$\int \underbrace{P_n(x)}_{u}\underbrace{e^{ax}dx}_{dv} \qquad \int \underbrace{ln\,ax}_{u}\cdot\underbrace{P_n(x)dx}_{dv}$$

$$\int \underbrace{P_n(x)}_{u}\underbrace{sin\,axdx}_{dv} \qquad \int \underbrace{arcsin\,ax}_{u}\cdot\underbrace{P_n(x)dx}_{dv}$$

$$\int \underbrace{P_n(x)}_{u}\underbrace{cos\,axdx}_{dv} \qquad \int \underbrace{arccos\,ax}_{u}\cdot\underbrace{P_n(x)dx}_{dv}$$

In this cases the polynomial $P_n(x)$ is selected as u	If the integral is a product of the logarithmic or inverse trigonometric function and a polynomial, then the mentioned function should be selected as u

Let's look at an example of the so called return integral, when in the result of integration by parts of the formula application, the received integral is similar to the initial one.

$$J=\int e^x\cos xdx=\left|\begin{matrix} e^x=u; \\ \cos xdx=dv; \\ du=e^xdx; \\ v=\sin x\end{matrix}\right|=e^x\sin x-\int e^x\sin xdx=$$

$$=\left|\begin{matrix} e^x=u_1 \\ \sin xdx=dv_1 \\ du_1=e^xdx \\ v_1=-\cos x\end{matrix}\right|=e^x\sin x-\left[e^x\left(-\cos x\right)+\int\cos xe^xdx\right]=$$

$$=e^x\sin x+e^x\cos x-\int e^x\cos xdx$$

then $J=e^x\left(\sin x+\cos x\right)-J$; $J=\dfrac{e^x}{2}\left(\sin x+\cos x\right)$

Method of integration by parts may be applied twice or three times if order of polynomial $P_n\left(x\right)$ equals two or three and so on.

Example $$\int\left(x^2+1\right)e^xdx=\left|\begin{matrix} x^2+1=u \\ e^xdx=dv \\ 2xdx=du \\ v=e^x\end{matrix}\right|=\left(x^2+1\right)e^x-\int e^xxdx=\left|\begin{matrix} x=u_1 \\ e^xdx=dv_1 \\ du_1dx \\ v_1=e^x\end{matrix}\right|=$$

$$=\left(x^2+1\right)e^x-2\left[xe^x-\int e^xdx\right]=\left(x^2+1\right)e^x-2xe^x+e^x+c.$$

Home task

1.	$\int\dfrac{e^{\sqrt{x-1}}}{\sqrt{x-1}}dx$	5.	$\int x^3e^{x^2}dx$
2.	$\int\dfrac{dx}{\left(x+4\right)\sqrt{x}}$	6.	$\int\cos\ln x\ dx$
3.	$\int x\ln 2xdx$	7.	$\int\left(2x^2+x-3\right)\sin xdx$
4.	$\int arctg\dfrac{x}{2}dx$	8.	$\int x^2\cos 3xdx$

4.4. Integration of the rational fractions

The rational fraction is the following expression:

$\frac{P_n(x)}{Q_m(x)}$, where $P_n(x)$ and $Q_m(x)$ − are polymonials of n and m degree.

Any rational fraction is called prorer if $n < m$ and improper if $n \geq m$

Any improper rational fraction can be represented as $\frac{P_n(x)}{Q_m(x)} = M(x) + \frac{P_k(x)}{Q_m(x)}$,

where $M(x)$ − an integral part, $\frac{P_k(x)}{Q_m(x)}$ − a proper rational fraction, $k < m$.

1. Integration of elementary (proper) rational fractions

I. $\int \frac{A}{x-a} dx = A \int \frac{d(x-a)}{x-a} = A\,ln(x-a) + C$; here A, $a = const$;

$\int \frac{3}{x-2} dx = 3 \int \frac{d(x-2)}{x-2} = 3\,ln(x-2) + C$.

II. $\int \frac{A}{(x-a)^m} dx = A \int (x-a)^{-m} d(x-a) = A \frac{(x-a)^{-m+1}}{1-m} + C$; m − an integer, more then 1.

$\int \frac{5}{(x-1)^3} = 5 \int (x-1)^{-3} d(x-1) = 5 \frac{(x-1)^{-2}}{-2} + C$;

III. $\int \frac{A}{x^2 + px + q} dx = A \int \frac{d\left(x + \frac{p}{2}\right)}{\underbrace{x^2 + 2\frac{p}{2}x + \frac{p^2}{4}}_{\left(x+\frac{p}{2}\right)^2} - \underbrace{\frac{p^2}{2} + q}_{\left(\sqrt{q - \frac{p^2}{4}}\right)^2}} =$

($p^2 - 4q < 0$, i.e. the denominator has no real roots)

Let’s extract in the denominator complete square and use the following formula

$\int \frac{du}{u^2 + a^2} = \frac{1}{a} arctg \frac{u}{a} + C$

$$= A\frac{1}{\sqrt{q-\frac{p^2}{4}}}arctg\frac{x+\frac{p}{2}}{\sqrt{q-\frac{p^2}{4}}}+C.$$

Example.

$$\int\frac{3}{x^2+2x+10}dx = 3\int\frac{dx}{x^2+2x+1+9} = 3\int\frac{d(x+1)}{(x+1)^2+3^2} = \frac{\not 3}{\not 3}arctg\frac{x+1}{3}+C.$$

Note $ax^2+bx+c = a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right) = a\left(x^2+px+q\right)$

where $\frac{b}{a}=p$; $\frac{c}{a}=q$

IV. $\int\frac{Ax+B}{x^2+px+q} = A\int\frac{x+\frac{B}{A}}{x^2+px+q}dx =$

1) Let’s extract in the numerator a derivative of the denominator
2) Let’s represent an integral as a sum of two integrals
3) In the first one we use the following formula $\int\frac{f'(x)dx}{f(x)}$,

$$=\frac{A}{2}\int\frac{2x+2\frac{B}{A}+p-p}{x^2+px+q}dx =$$

$$\frac{A}{2}\int\frac{2x+p}{x^2+px+q}dx+\frac{A}{2}\left(2\frac{B}{A}-p\right)\int\frac{dx}{x^2+2\frac{p}{2}x+\frac{p^2}{4}-\frac{p^2}{4}+q} =$$

$$=\frac{A}{2}ln\left(x^2+px+q\right)+\left(B-\frac{Ap}{2}\right)\frac{2}{\sqrt{4q-p^2}}arctg\frac{2x+p}{\sqrt{4q-p^2}}+C;$$

Example.

$$\int\frac{5x+1}{x^2-2x+5}dx = \frac{5}{2}\int\frac{2x+\frac{2}{5}-2+2}{x^2-2x+5}dx =$$

$\langle 2x-2\rangle$

$$=\frac{5}{2}\int\frac{2x-2}{x^2-2x+5}dx+\frac{\not 5}{2}\left(\frac{2}{\not 5}+\frac{10}{\not 5}\right)\int\frac{d(x-1)}{\underset{(x-1)^2+2^2}{x^2-2x+1+4}} =$$

$$=\frac{5}{2}ln\left(x^2-2x+5\right)+\frac{3}{6}\cdot\frac{1}{2}arctg\frac{x-1}{2}+C;$$

$x^2-2x+5>0$, $D=4-20=-16<0$.

2. Integration of rational fraction by disclosure on the simplest with the help of undetermined coefficients.

Step 1. Let's check if the rational fraction is proper. If it is improper then let's extract an integral part dividing polynomial by polynomial in column

$$\int\frac{P_n(x)}{Q_m(x)}dx=\int\left(M(x)+\frac{P_k(x)}{Q_m(x)}\right)dx$$

$(n\geq m)$ $\qquad\qquad$ $(k<m)$

Step 2. Let's divide it into two integrals $\int\frac{P_n(x)}{Q_m(x)}dx=\int M(x)dx+\int\frac{P_k(x)}{Q_m(x)}dx$ and consider the second one

Step 3. Let's represent proper rational fraction $\frac{P_k(x)}{Q_m(x)}$ as a sum of the simplest rational fraction for that we expand the denominator into linear and quadratic multipliers

$$\frac{P_k(x)}{Q_m(x)}=\frac{A_1}{(x-a)^{m_1}}+\frac{A_2}{(x-a)^{m_1-1}}+\ldots+\frac{A_{m_1}}{x-a}+\ldots+$$
$$+\frac{B_1x+C_1}{\left(x^2+px+q\right)^{n_1}}+\frac{B_2x+C_2}{\left(x^2+px+q\right)^{n_1-1}}+\ldots+\frac{B_{n_1}x+C_{n_1}}{x^2+px+q}+\ldots; \qquad (*)$$

Step 4. Let's define undetermined coefficients A,B,C for that we reduce the right part of the equality (*) to the common denominator and consider only the nominator. This procedure is called conversion of the denominator.

Let's equate the coefficients on the left and on the right at equal degrees x.

We have a system of the linear algebraic equations. Considering it let's find undetermined coefficients as an equality (*) must be carried out at any x, then undetermined coefficients can be found in some other way namely giving x different values advisable equal to real roots of the denominator.

Both of these methods can be combined. In the result an integral of the improper rational fraction will be reduced to the integral of the polynomial and of the simplest rational fractions.

Let's consider three cases:

Case 1. Denominator has only real different roots

$$\int \frac{2x^3+x^2-2}{x^3-x}dx=\int\left(2+\frac{x^2+2x-2}{x^3-x}\right)dx=$$

The integral is improper rational fraction. Let's extract an integral part

$$\begin{array}{l|l} -\,2x^3+x^2+0x-2 & \\ \quad 2x^3 \qquad\quad -2x & x^3-x \\ \hline \quad x^2+2x-2 & 2 - \text{the resul of division} - \text{an integral part} \end{array}$$

a remainder

$$=2\int dx+\int\frac{x^2+2x-2}{x(x-1)(x+1)}dx=(**)$$

Let's write down a proper rational fraction

$$\frac{x^2+2x-2}{x^3-x}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}$$

Let's free from the denominator

$$x^2+2x-2=A(x-1)(x+1)+Bx(x+1)+Cx(x-1)$$

Equating the coefficients at equal degrees x on the left and on the right we obtain a system of equations

$$\left.\begin{array}{l|l} x^2 & 1=A+B+C \\ x^1 & 2=B-C \\ x^0 & -2=-A \end{array}\right\} \quad A=2; \qquad \begin{cases} B+C=-1 \\ B-C=2 \end{cases} +$$

$$2B=1; \quad B=\frac{1}{2};$$

$$C=-\frac{3}{2};$$

The simple way for finding of undetermined coefficients is giving to x the meaning equal to the real roots

$$\left.\begin{array}{l} x=0 \\ x=1 \\ x=-1 \end{array}\right| \begin{array}{l} -2=-A;\ A=2; \\ 1+\not{2}-\not{2}=A\cdot 0+2B+C\cdot 0;\ B=\frac{1}{2}; \\ 1-2-2=A\cdot 0+B\cdot 0+2C;\ C=-\frac{3}{2}; \end{array}$$

$$(**)=2\int dx+2\int\frac{dx}{x}+\frac{1}{2}\int\frac{dx}{x-1}-\frac{3}{2}\int\frac{dx}{x+1}=$$
$$=2x+2\,ln|x|+\frac{1}{2}ln|x-1|-\frac{3}{2}ln|x+1|+C;$$

Case 2. Denominator has real roots and some of them are multiple.

$$\int\frac{x^2+1}{x(x-1)^3}dx=-\int\frac{dx}{x}+2\int(x-1)^{-3}dx+\int\frac{dx}{x-1}=$$

$$\frac{x^2+1}{x(x-1)^3}=\frac{A}{x}+\frac{B}{(x-1)^3}+\frac{C}{(x-1)^2}+\frac{D}{x-1};$$

$$x^2+1=A(x-1)^3+Bx+Cx(x-1)+(x-1)^2$$

$$\left.\begin{array}{l} x=0 \\ x=1 \\ x^3 \\ x^1 \end{array}\right| \begin{array}{l} 1=-A;\quad A=-1; \\ 2=B;\quad B=2; \\ 0=A+D;\quad D=-A=1; \\ 0=+3A+B-C+D;\quad C=-3+2+1=0 \end{array}$$

$$=-ln|x|+\frac{\not{2}\,(x-1)^{-2}}{-\not{2}}+ln|x-1|+ln\,C=ln\frac{C|x-1|}{|x|}-\frac{1}{(x-1)^2}.$$

Case 3. Among the roots of denominator there are simple complex conjugated roots, i.e. expansion of denominator contains quadratic non-recurring multipliers.

$$\int\frac{3dx}{x^3-1}=\int\frac{dx}{(x-1)(x-1)(x^2+x+1)}=\int\frac{dx}{x-1}-\int\frac{x+2}{x^2+x+1}dx=$$

$$2x+1$$

$$\frac{3}{(x-1)(x^2+x+1)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}$$

$$3=A(x^2+x+1)+(Bx+C)(x-1)$$

$$\begin{array}{l|l} x=1 & 3=A\cdot 3+\left(Bx+C\right)\cdot 0;\quad A=1; \\ x^2 & 0=A+B;\quad B=-A=-1; \\ x^0 & 3=A-C;\quad C=A-3=-2; \end{array}$$

$$=\int\frac{dx}{x-1}-\frac{1}{2}\int\frac{2x+4}{x^2+x+1}=\int\frac{dx}{x-1}-\frac{1}{2}\int\frac{2x+1+3}{x^2+x+1}dx=$$

$$\int\frac{dx}{x-1}-\frac{1}{2}\int\frac{2x+1}{x^2+x+1}dx-\frac{3}{2}\int\frac{dx}{x^2+2\frac{1}{2}x+\frac{1}{4}-\frac{1}{4}+\frac{4}{4}}=$$

$$\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2$$

$$=ln\left|x-1\right|-\frac{1}{2}ln\left(x^2+x+1\right)-\frac{3}{\not{2}}\cdot\frac{\not{2}}{\sqrt{3}}arctg\frac{2x+1}{\sqrt{3}}+C;$$

Home task

1. $\int\frac{x+1}{x\left(x-2\right)}dx$

2. $\int\frac{2x^2+x-1}{\left(x+1\right)^2\left(x-2\right)^2}dx$

3. $\int\frac{3x^2-1}{\left(x^2+4\right)\left(x-2\right)}dx$

4. $\int\frac{x^2-1}{\left(x^2+1\right)\left(x^2+4\right)}dx$

5. $\int\frac{x^2+1}{x^4+4}dx$, note $\left[x^4+1=\left(x^2+2\right)^2-4x^2\right]$

4.5. Integration of the following irrationalities

a. $\int\frac{dx}{\sqrt{ax^2+bx+c}}$ b. $\int\frac{Ax+B}{\sqrt{ax^2+bx+c}}dx$

Proceeding similarly to paragraphs III and IV the integration of the simplest rational fractions.

a) $\int\frac{dx}{\sqrt{ax^2+bx+c}}$

Let's extract at the root the complete square and depending on a, b, c signs we use the following formulae $\int\frac{du}{\sqrt{u^2\pm\lambda}}$ or $\int\frac{du}{\sqrt{a_1^2-u^2}}$

Example. $\int\frac{dx}{\sqrt{x^2+2x+2}}=\int\frac{dx}{\sqrt{x^2+2x+1+1}}=\int\frac{d(x+1)}{\sqrt{(x+1)^2+1}}=$

$=ln\left|x+1+\sqrt{x^2+2x+2}\right|+C$

Example. $\int\frac{dx}{\sqrt{6+4x-2x^2}}=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{-(x^2-2x+1-4)}}=$

$=\frac{1}{\sqrt{2}}\int\frac{d(x-1)}{\sqrt{2^2-(x-1)^2}}=\frac{1}{\sqrt{2}}arcsin\frac{x-1}{2}+C;$

Let's extract in the nominator derivative of subradical expression and divide it into two integrals. In the first we will use the following formula $\int\frac{f'(x)}{\sqrt{f(x)}}dx$ and in the second − see a).

Example. $\int\frac{x-1}{\sqrt{6x-x^2-8}}=-\frac{1}{2}\int\frac{-2x+2+6-6}{\sqrt{6x-x^2-8}}dx=$

$=-\frac{1}{2}\int\frac{-2x+6}{\sqrt{-x^2+6x-8}}dx+\frac{4}{2}\int\frac{d(x-3)}{\sqrt{-(x^2-6x+9-9+8)}}=$

$-(x-3)^2+1$

$=-\frac{1}{\cancel{2}}\cdot\cancel{2}\sqrt{-x^2+6x-8}+2\,arcsin(x-3)+C.$

Home task

1. $\int\frac{dx}{\sqrt{x^2-x-1}}$

2. $\int\frac{dx}{\sqrt{-x^2-2x+3}}$

3. $\int\frac{x^2+1}{\sqrt{2x-x^2}}dx$

4. $\int\frac{3x-1}{\sqrt{x^2+x+1}}dx$

4.6. Integration of Trigonometric Functions

1. Integrals of the kind $\int sin^n x\, cos^m x dx$

Let's consider two cases

1) If at least one of the exponent m or n is an odd number.

If n is an odd number then we should use the substitution $cos\, x = t$, if m is an odd number then $sin\, x = t$.

Let's consider examples

1. $\int sin^3 x\, cos^2 x dx = \begin{vmatrix} cos\, x = t \\ -sin\, x dx = dt \end{vmatrix} =$

$= \int sin^2 x\, cos^2 x\, sin\, x dx = -\int \left(1-t^2\right) t^2 dt =$

$= \int \left(t^4 - t^2\right) dt = \frac{t^5}{5} - \frac{t^3}{3} + c = \frac{cos^5 x}{5} - \frac{cos^3 x}{3} + c;$

2. $\int \frac{cos^3 x}{sin\, x} dx = \int \frac{cos^2 x\, cos\, x dx}{sin\, x} = \begin{vmatrix} sin\, x = t \\ cos\, x dx = dt \end{vmatrix} =$

$= \int \frac{1-t^2}{t} dt = \int \left(\frac{1}{t} - t\right) dt = ln|t| - \frac{t^2}{2} + c = ln|sin\, x| - \frac{sin^2 x}{2} + c;$

2) Both m and n are even numbers then we should use the following formulae:

$$sin\, x\, cos\, x = \frac{1}{2} sin\, 2x; \quad sin^2 x = \frac{1-cos\, 2x}{2}; \quad cos^2 x = \frac{1+cos\, 2x}{2}.$$

1. $\int sin^2 x\, cos^2 x dx = \int \left(\frac{1}{2} sin\, 2x\right)^2 dx = \frac{1}{4} \int sin^2 2x dx =$

$= \frac{1}{4} \int \frac{1-cos\, 4x}{2} dx = \frac{1}{8} \left(x - \frac{1}{4} sin\, 4x\right) + c.$

2. $\int sin^4 x dx = \int \left(sin^2 x\right)^2 dx = \int \left(\frac{1-cos\, 2x}{2}\right)^2 dx =$

$= \frac{1}{4} \int \left(1 - 2 cos\, 2x + cos^2 2x\right) dx = \frac{1}{4} \int \left(1 - 2 cos\, 2x + \frac{1+cos\, 4x}{2}\right) dx =$

$= \frac{1}{4} \int \left(\frac{3}{2} - 2 cos\, 2x + \frac{1}{2} cos\, 4x\right) dx = \frac{1}{4} \left[\frac{3}{2} x = sin\, 2x + \frac{1}{8} sin\, 4x\right] + c =$

$$=\frac{3}{8}x-\frac{1}{4}sin\,2x+\frac{1}{32}sin\,4x+c;$$

2. Integrals of the kind $\int tg^m xdx$, $\left(\int ctg^m xdx\right)$,

where m − is a whole number.

We decrease the power of tangent or cotangent applying in series the formulae

$$tg^2x=\frac{1}{cos^2 x}-1,\; ctg^2x=\frac{1}{sin^2 x}-1.$$

Example. $\int tg^3xdx=\int tg^2xtgxdx=\int tgx\left(\frac{1}{cos^2 x}-1\right)dx=$

$$=\int tgxdtgx-\int tgxdx=\left|\frac{1}{cos^2 x}dx=dtgx\right|=\frac{tg^2x}{2}+ln\left|cos\,x\right|+c$$

Example $\int ctg^4xdx=\int ctg^2x\cdot ctg^2xdx=\int ctg^2x\left(\frac{1}{sin^2 x}-1\right)dx=$

$$=\left|\frac{1}{sin^2 x}dx=-dctgx\right|=-\int ctg^2xdctgx-\int\left(\frac{1}{sin^2 x}-1\right)dx=$$

$$=-\int ctg^2xdctgx+\int dctgx+\int dx=-\frac{ctg^3x}{3}+ctgx+x+c;$$

3. Integrals of the kind $\int tg^m x\frac{dx}{cos^n x}$, $\left(\int ctg^m x\frac{dx}{sin^n x}\right)$,

where n is an even number.

Here we apply the following formulas $\frac{1}{cos^2 x}=tg^2x+1$, $\frac{1}{sin^2 x}=ctg^2x+1$

Example. $\int tg^3x\frac{dx}{cos^4 x}=\int tg^3x\frac{1}{cos^2 x}\cdot\frac{1}{cos^2 x}dx=$

$$=\int tg^3x\left(tg^2x+1\right)dtgx=\int\left(tg^5x+tg^3x\right)dtgx==\frac{tg^6x}{6}+\frac{tg^4x}{4}+c.$$

Example. $\int\frac{dx}{sin^6 x}=\int\left(\frac{1}{sin^2 x}\right)^2\frac{1}{sin^2 x}dx=-\int\left(ctg^2x+1\right)^2 dctgx=$

$$=-\int\left(ctg^4x+2ctg^2x+1\right)dctgx=-\frac{ctg^5x}{5}-\frac{2}{3}ctg^3x+ctgx+c;$$

4. Integrals of the kind: $\int sin\,mx\cdot cos\,nxdx$, $\int sin\,mx\cdot sin\,nxdx$, $\int cos\,mx\cdot cos\,nxdx$.

Let's present the product of trigonometric functions as a sum with the help of the following formulae:

$$sin\,mx\cdot cos\,nx=\frac{1}{2}\left[sin\left(m+n\right)x+sin\left(m-n\right)x\right]$$

$$cos\,mx\cdot cos\,nx=\frac{1}{2}\left[cos\left(m+n\right)x+cos\left(m-n\right)x\right]$$

$$sin\,mx\cdot sin\,nx=\frac{1}{2}\left[cos\left(m-n\right)x-cos\left(m+n\right)x\right]$$

Example. $\int sin\,3x\cdot cos\,2xdx=\frac{1}{2}\int\left(sin\,5x+sin\,x\right)dx=$

$=-\frac{1}{2}\cdot\frac{1}{5}cos\,5x-\frac{1}{2}cos\,x+c$.

5. <u>Integral of the kind</u> $\int R\left(sin\,x,cos\,x\right)dx$

1) Here $sin\,x$, $cos\,x$ are the parts of integration function in the first power not multiplying with each other.

Then let's use the general trigonometric substitution:

$$tg\frac{x}{2}=t,\qquad sin\,x=\frac{2t}{1+t^2},\qquad cos\,x=\frac{1-t^2}{1+t^2};\qquad dx=\frac{2dt}{1+t^2}.$$

2) If $R\left(sin\,x,cos\,x\right)$ is an odd function regarding $sin\,x$, i.e.

$R\left(-sin\,x,cos\,x\right)=-R\left(sin\,x,cos\,x\right)$, then we should use the substitution $cos\,x=t$

3) If $R\left(sin\,x,cos\,x\right)$ is an odd function regarding $cos\,x$, i.e.

$R\left(-sin\,x,cos\,x\right)=-R\left(sin\,x,cos\,x\right)$

then we should use the substitution $sin\,x=t$.

4) If $R\left(sin\,x,cos\,x\right)$ is an even function regarding $sin\,x$ and $cos\,x$, i.e. $R\left(-sin\,x,-cos\,x\right)=R\left(sin\,x,cos\,x\right)$, then we should use the substitution: $tgx=t$,

then $sin\,x=\frac{t}{\sqrt{1+t^2}}$; $cos\,x=\frac{1}{\sqrt{1+t^2}}$; $dx=\frac{dt}{1+t^2}$.

Example . $\int\frac{dt}{sin\,x+1}=\left|\begin{matrix}tg\frac{x}{2}=t,dx=\frac{2dt}{1+t^2}\\ sin\,x=\frac{2t}{1+t^2}\end{matrix}\right|=$

$$\int \frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2}+1} = 2\int \frac{\frac{dt}{\cancel{1+t^2}}}{\frac{2t+1+t^2}{\cancel{1+t^2}}} = 2\int \frac{dt}{(t+1)^2} = \frac{-2}{t+1}+c = \frac{-2}{tg\frac{x}{2}+1}+c;$$

Example. $\int \frac{cos\,x - cos^3 x}{sin^2 x - sin^4 x}dx = \left| \begin{matrix} sin\,x = t \\ cos\,xdx = dt \end{matrix} \right| = \int \frac{\left(1-cos^2 x\right)cos\,xdx}{sin^2 x - sin^4 x} =$

$$= \int \frac{t^2 dt}{t^2\left(1-t^2\right)} = -\int \frac{dt}{t^2-1} = -\frac{1}{2}ln\left|\frac{t-1}{t+1}\right| + c = \frac{1}{2}ln\left|\frac{t+1}{t-1}\right| + c = ln\sqrt{\frac{sin\,x+1}{sin\,x-1}} + c;$$

Example. $\int \frac{dx}{2\,sin\,x\,cos\,x - cos^2 x} = \left| \begin{matrix} tgx = t & cos\,x = \frac{1}{\sqrt{1+t^2}} \\ sin\,x = \frac{t}{\sqrt{1+t^2}}; & dx = \frac{dt}{1+t^2} \end{matrix} \right| =$

$$= \int \frac{\frac{dt}{1+t^2}}{2\frac{t}{\sqrt{1+t^2}}\frac{1}{\sqrt{1+t^2}} - \frac{1}{1+t^2}} = \int \frac{dt}{2t-1} = \frac{1}{2}ln|2t-1| + c = ln\sqrt{2tgx-1} + c;$$

4.7. Trigonometric substitutions

Let's consider the integrals where R is a rational function

1. $\int R\left(x, \sqrt{a^2-x^2}\right)dx$, let's use a substitution here

$x = a\,sin\,t \quad \left(x = a\,cos\,t\right)$ then $dx = a\,cos\,tdt$, $\sqrt{a^2 - a^2 sin^2 t} = a\,cos\,t$.

2. $\int R\left(x, \sqrt{a^2+x^2}\right)dx$ substitution $x = atgt\left(x = actgt\right)$

$$dx = \frac{adt}{cos^2 t}; \qquad \sqrt{a^2 + a^2\frac{sin^2 t}{cos^2 t}} = a\sqrt{\frac{cos^2 t + sin^2 t}{cos^2 t}} = \frac{a}{cos\,t};$$

3. $\int R\left(x, \sqrt{x^2-a^2}\right)dx$ substitution $x = \frac{a}{sin\,t}\left(x = \frac{a}{cos\,t}\right)$

$$dx = -a\frac{cos\,t}{sin^2 t}dt; \quad \sqrt{\frac{a^2}{sin^2 t} - a^2} = a\sqrt{\frac{1-sin^2 t}{sin^2 t}} = a\frac{cos\,t}{sin\,t}$$

Let's consider examples for each case

1. $$\int\frac{\sqrt{a^2-x^2}}{x^2}dx=\left|\begin{matrix}x=a\sin t\\ dx=a\cos t dt\end{matrix}\right|=\int\frac{\sqrt{a^2-a^2\sin^2 t}}{a^2\sin^2 t}a\cos t dt=$$

$$=\int\frac{\cos^2 t}{\sin^2 t}dt=\int\left(\frac{1}{\sin^2 t}-1\right)dt=-ctgt-t+c=$$

$$=\left|\begin{matrix}\sin t=\frac{x}{a};t=\arcsin\frac{x}{a}\\ \cos t=\sqrt{1-\sin^2 t}=\sqrt{1-\frac{x^2}{a^2}}\end{matrix}\right|=-\frac{\sqrt{a^2-x^2}\cdot \not a}{\not a\cdot x}-\arcsin\frac{x}{a}+c.$$

2. $$\int\frac{dx}{x\sqrt{a^2+x^2}}=\left|\begin{matrix}x=atgt\\ dx=\frac{a}{\cos^2 t}dt\end{matrix}\right|=\int\frac{\not a dt}{\not a tgt\sqrt{a^2+a^2tg^2t}\cos^2 t}=$$

$$=\frac{1}{a}\int\frac{\cos t dt}{\sin t\frac{1}{\cos t}\cos^2 t}=\frac{1}{a}\int\frac{dt}{\sin t}=\frac{1}{a}ln\left|\frac{1}{\sin t}-ctgt\right|+c=$$

$$\left|\begin{matrix}ctgt=\frac{a}{x}\\ \frac{1}{\sin t}=\sqrt{1+ctg^2t}=\frac{\sqrt{a^2+x^2}}{x}\end{matrix}\right|=\frac{1}{a}ln\left|\frac{\sqrt{a^2+x^2}-a}{x}\right|+c.$$

3. $$\int\frac{\not x^{\not 2}dx}{\sqrt{x^2-a^2}}=\left|\begin{matrix}x=\frac{a}{\sin t}\\ dx=-a\frac{\cos t}{\sin^2 t}dt\end{matrix}\right|=-\int\frac{\frac{a}{\sin t}\cdot\frac{a\cos t}{\sin^2 t}dt}{\sqrt{\frac{a^2}{\sin^2 t}-a^2}}=$$

$$=-a\int\frac{\sin^{\not 2} t\cos t dt}{\sin^3 t\cos t}=-a\int\frac{dt}{\sin^2 t}=actgt+c=\left|\begin{matrix}\sin t=\frac{a}{x}\\ \cos t=\sqrt{1-\frac{a^2}{x^2}}\end{matrix}\right|=$$

$$=a\frac{\sqrt{x^2-a^2}\cdot x}{x\cdot a}+c=\sqrt{x^2-a^2}+c.$$

Home task

1. $\int \frac{dx}{3+5\sin x+3\cos x}$

2. $\int \frac{dx}{1-\cos x}$

3. $\int \frac{\cos^2 xdx}{\sin^2 x+4\sin x\cos x}$

4. $\int \frac{\sin 2xdx}{\cos^3 x-\sin^2 x-1}$

5. $\int \sin^3 xdx$

6. $\int \frac{\cos^5 x}{\sin x}dx$

7. $\int \sin 3x\sin xdx$

8. $\int \cos 3x\cos 2xdx$

9. $\int ctg^4 \frac{x}{2}dx$

5. DEFINITE INTEGRAL

5.1. Definition. Properties

Let function be determined on the interval [a, b]. Let's divide the interval [a, b] into n subintervals by points $a=x_0<x_1<x_2<...<x_n=b$.

Let's choose on each elementary segment $[x_{k-1}, x_k]$ some point ξ_κ and find the length of each segment $\Delta x_k=x_k-x_{k-1}$. Let's compound an integral sum

$$\sum_{k=1}^{n} f(\xi_k)\Delta x_k=f(\xi_1)\Delta x_1+f(\xi_2)\Delta x_2+...+f(\xi_n)\Delta x_n.$$

The definite integral of the function $f(x)$ on the interval [a, b] is the limit of the integral sum when the length of the largest of the elementary segments tends to zero

$$\int_a^b f(x)dx=\lim_{max\Delta_k\to 0}\sum_{k=1}^{n} f(\xi_k)\Delta x_k.$$

The theorem of the definite integral existence. If function $f(x)$ be continuous on the interval [a, b] then the limit of integral sum exists and doesn't depend on the method of segment division [a, b] into elementary and chosen points ξ_κ.

If $f(x)>0$ on the interval [a, b] then the definite integral $\int_a^b f(x)dx$ in geometrical meaning is the area of the curvilinear trapezoid with the base restricted by the lines $y = f(x)$, $x = a$, $x = b$, $y = 0$ (Fig. 8).

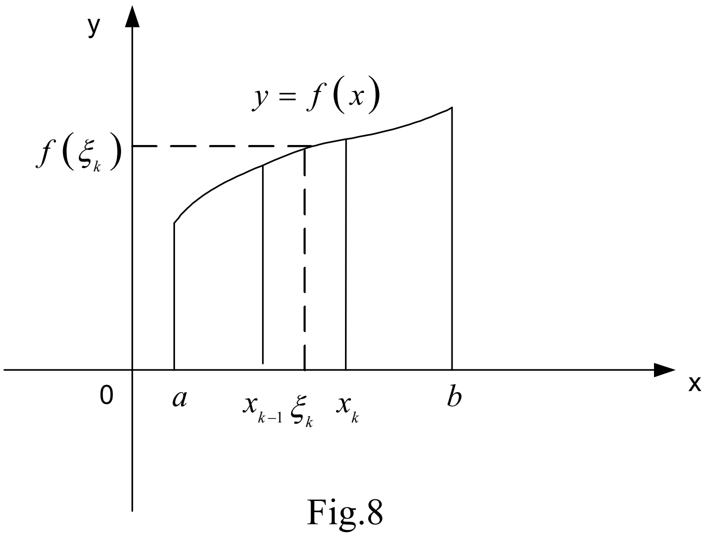


Fig.8

<u>Basic properties of the definite integral</u>

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$.

2. $\int_a^a f(x)dx = 0$.

3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, $c \in$[a, b].

4. $\int_a^b [f_1(x) \pm f_2(x)]dx = \int_a^b f_1(x)dx \pm \int_a^b f_2(x)dx$.

5. $\int_a^b \lambda f(x)dx = \lambda \int_a^b f(x)dx$, $\lambda = const$

6. Estimation of the definite integral. If m and M are the least and the most meanings of the function $f(x)$ on the interval [a, b] then

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a).$$

As $f(x)$ is continuous on the interval [a, b] then there is such $f_{mean}(x)$ that

$$f_{mean}(x) = \frac{1}{b-a} \int_a^b f(x)dx$$

<u>Rules of definite integral calculation</u>

1. The formula of Newton-Leibnitz

$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$, where $F(x)$ − antiderivative for $f(x)$ i.e. $F'(x) = f(x)$

2. Integration by parts

$\int_a^b udv = uv\Big|_a^b - \int_a^b vdu$, where $u = u(x)$, $v = v(x)$ − continuous differential functions on the interval [a, b].

3. The substitution of variable

$\int_a^b f(x)dx = \int_\alpha^\beta f[\varphi(t)]\varphi'(t)dt$, where $x = \varphi(t)$ − the function is continuous together with its derivative $\varphi'(t)$ on the interval [α, β].

$$\alpha \le t \le \beta, \quad a = \varphi(\alpha), \quad b = \varphi(\beta),$$

$f[\varphi(t)]$ − the function is continuous on the interval [α, β].

4. If $f(x)$ − the odd function i.e. 1) the area of definition is symmetric regarding zero and 2) $f(-x) = -f(x)$, then $\int_{-a}^{a} f(x) = 0$.

If $f(x)$ is even function i.e. 1) the area of definition is symmetric regarding zero and 2) $f(-x) = f(x)$, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$.

Let's consider examples of the definite integrals calculation.

Example . $\int_{\pi/6}^{\pi/3} \frac{dx}{sin^2 x} = -ctgx\Big|_{\pi/6}^{\pi/3} = -\left(ctg\frac{\pi}{3} - ctg\frac{\pi}{6}\right) =$

$= -\left(\frac{1}{\sqrt{3}} - \frac{\sqrt{3}}{1}\right) = \sqrt{3} - \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3}.$

Example. $\int_1^e ln\, xdx = \begin{vmatrix} ln\, x = u; & du = \frac{1}{x}dx \\ dx = dv & v = x \end{vmatrix} = x\, ln\, x\Big|_1^e - \int_1^e \not{x}\frac{1}{\not{x}}dx = x\, ln\, x\Big|_1^e - x\Big|_1^e =$

$= e\, ln\, e - 1 ln 1 - (e-1) = \not{e} - \not{e} + 1 = 1.$

Example. Let's estimate $\int_0^{\pi/2} \frac{dx}{1+cos^2 x}$ as $0 \le cos^2 x \le 1$ then $m = \frac{1}{1+1} = \frac{1}{2}$;

$$M=\frac{1}{1+0}=1;\quad m(b-a)=\frac{1}{2}\left(\frac{\pi}{2}-0\right)\le\int_0^{\pi/2}\frac{dx}{1+cos^2x}\le 1\left(\frac{\pi}{2}-0\right)=M(b-a)$$

$$0,7854=\frac{3,1416}{4}=\frac{\pi}{4}\le\int_0^{\pi/2}\frac{dx}{1+cos^2x}\le\frac{\pi}{2}=\frac{3,1416}{2}=1,5708$$

Example. $\int_0^R\sqrt{R^2-x^2}\,dx=\left|\begin{array}{l}x=R\,sin\,t\\ dx=R\,cos\,tdt\\ x_1=0;\quad 0=R\,sin\,t_1;\quad t_1=0\\ x_2=R;\quad R=R\,sin\,t_2;\quad t_2=\dfrac{\pi}{2}\end{array}\right|=$

$$=\int_0^{\pi/2}\sqrt{R^2-R^2sin^2t}\cdot R\,cos\,tdt=R^2\int_0^{\pi/2}cos^2tdt=$$

$$=\frac{R^2}{2}\int_0^{\pi/2}(1+cos\,2t)dt=\frac{R^2}{2}\left[t+\frac{1}{2}sin\,2t\right]\Bigg|_0^{\pi/2}=\frac{R^2}{2}\left[\frac{\pi}{2}-0+\frac{1}{2}(sin\,\pi-sin\,0)\right]=$$

$$=\frac{\pi R^2}{4}.$$

Notice that is an area of the circle quarter.

Example. $\int_{-1}^{1}x^2\,arcsin\,xdx=0\Big|$ The integrant is odd as the product of even on odd, therefore the definite integral in the symmetric limits is equal to zero.

Example. $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\frac{dx}{cos^2x}=2\int_0^{\pi/3}\frac{dx}{cos^2x}=2tgx\Big|_0^{\pi/3}=2\left(tg\frac{\pi}{3}-tg0\right)=$

$$=2tg\frac{\pi}{3}=2\sqrt{3}.$$

The integrand is even.

5.2. Improper integrals

Improper integrals − are

1) integrals with infinite limits
2) integrals of unbounded functions

1) According to the definition

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx; \quad \int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx;$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{\substack{a\to-\infty \\ b\to\infty}}\int_a^b f(x)\,dx$$

If these limits exist and are finite then improper integrals concentrate or exist but if the limits do not exist or they are equal to infinity then improper integrals do not exist.

2) If the function $f(x)$ has an infinite break in c point on the interval [a, b] and is continuous for all the rest $x\in$[a, b], then

$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0}\int_a^{c-\varepsilon} f(x)\,dx + \lim_{\delta\to 0}\int_{c+\delta}^{b} f(x)\,dx$$

Improper integral converges or exists if both limits in the right part of this equality exist and diverge or does not exist if at least one of them does not exist.

Signs of comparison

1) If the function $f(x)$ and $\varphi(x)$ are defined for all $x \ge a$ and are integrated on the interval [a, A] where $A \ge a$, and if $0 \le f(x) \le \varphi(x)$ for all $x \ge a$, then from the convergence of the integral $\int_a^{\infty}\varphi(x)\,dx$ there follows the convergence of $\int_a^{\infty} f(x)\,dx$, where $\int_a^{\infty} f(x)\,dx \le \int_a^{\infty}\varphi(x)\,dx$.

2) a) When $x\to\infty$ $$\int_a^{\infty}\frac{dx}{x^p} = \begin{cases}\text{converges when } p>1\\ \text{diverges when } p\le 1\end{cases}$$

b) When $x\in[a,b)$ $$\int_a^b \frac{dx}{(b-x)^p} = \begin{cases}\text{converges when } p<1\\ \text{diverges when } p\ge 1\end{cases}$$

Example. $\int_2^{\infty}\frac{dx}{x^2} = \lim_{b\to\infty}\int_2^b\frac{dx}{x^2} = \lim_{b\to\infty} -\frac{1}{x}\Big|_2^b = -\lim_{b\to\infty}\left(\frac{1}{b}-\frac{1}{2}\right) = \frac{1}{2}$ − converges,

that is confirmed by the sign 2 a).

Example. $\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = 2\int_0^{\infty}\frac{dx}{1+x^2} = 2\lim_{b\to\infty}\int_0^b\frac{dx}{1+x^2} =$

$= 2\lim_{b\to\infty} arctgx\Big|_0^b = 2\lim_{b\to\infty}(arctgb - arctg0) = 2\cdot\left(\frac{\pi}{2}-0\right) = \pi$ − converges.

Example. $\int_1^{\infty}\frac{ln(x+1)}{x+1}dx=\lim_{b\to\infty}\int_1^b ln(x+1)\,d\,ln(x+1)=$

$$=\lim_{b\to\infty} ln^2(x+1)\Big|_1^b=\lim_{b\to\infty}\Bigg(\underbrace{ln^2(b+1)}_{\searrow\infty}-ln^2 2\Bigg)=\infty$$ – diverges.

5.3. Applications of the definite integral

Calculation of plane figure area

The area of the curvilinear trapezoid bounded by the curve $y=f(x)$ $[f(x)\ge 0]$ the straight lines $x=a$, $x=b$ and the interval [a, b] of the X-axis is calculated according to the formula $S=\int_a^b f(x)dx$.

The area of a figure bounded by the curves $y=f_1(x)$ and $y=f_2(x)$, $[f_1(x)\le f_2(x)]$ and the straight lines $x=a$, $x=b$ are calculated using the formula

$$S=\int_a^b\left[f_2(x)-f_1(x)\right]dx.$$

If curve is given by the parametric equations $x=x(t)$, $y=y(t)$, then the area of the curvilinear trapezoid bounded by this curve the straight lines $x=a$, $x=b$ and the interval [a, b] of the X-axis is calculated using the formula

$$S=\int_{t_1}^{t_2} y(t)x'(t)dt,$$

where t_1 and t_2 are defined out of the equations $a=x(t_1)$; $b=x(t_2)$, $y(t)\ge 0$ if $t_1\le t\le t_2$.

The area of the curvilinear sector bounded by the curve given in polar coordinates with the help of the equation $\rho=\rho(\varphi)$ and with two polar radiuses $\varphi=\alpha$, $\varphi=\beta$ $(\alpha<\beta)$ is expressed by the formula

$$S=\frac{1}{2}\int_{\alpha}^{\beta}\rho^2 d\varphi.$$

Example. To calculate the area of a figure bounded by the lines: $y = x^2$; $y = x + 2$

$y_1 = x^2$; $y_2 = x + 2$, $(y_2 \geq y_1)$.

Solution. Let's find the points of intersection $x^2 = x + 2$; $x^2 - x - 2 = 0$; $x_{12} = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = \begin{cases} 2 \\ -1 \end{cases}$.

Then $S = \int_{-1}^{2} \left(x + 2 - x^2 \right) dx = \frac{x^2}{2} \Big|_{-1}^{2} + 2x \Big|_{-1}^{2} - \frac{x^3}{3} \Big|_{-1}^{2} =$

$= \frac{1}{2}(4-1) + 2(2+1) - \frac{1}{3}(8+1) = 4{,}5$ (square units)

Example. To calculate the area bounded by one arc of the cycloid $x = a(t - sin\,t)$, $y = a(1 - cost)$ and by the X-axis.

Solution. t changes from 0 to 2π. $dx = a(1 - cost)dt$

$$S = \int_{0}^{2\pi} a^2 (1 - cost)^2 \, dt = a^2 \int_{0}^{2\pi} \left(1 - 2cos\,t + cos^2 t\right) dt =$$

$$a^2 \int_{0}^{2\pi} \left(1 - 2cos\,t + \frac{1 + cos\,2t}{2} \right) dt = a^2 \left[\frac{3}{2}t - 2\,sin\,t + \frac{1}{4} sin\,2t \right] \int_{0}^{2\pi} =$$

$$a^2 \left[\frac{3}{2} 2\pi - 2(sin\,2\pi - sin\,0) + \frac{1}{4}(sin\,4\pi - sin\,0) \right] = 3\pi a^2 \,(square\ units)$$

Plane curve arc length calculation

If a curve $y = f(x)$ on the interval [a, b] has continuous derivative then the arc length of this curve is

$$L = \int_{a}^{b} \sqrt{1 + (y')^2}\, dx$$

In the case of parametric representation of the curve $x = x(t)$, $y = y(t)$, where $x(t)$, $y(t)$ are continuously differentiating function the arc length of the curve changing the parameter t of t_1 to t_2 is calculated according to the formula

$$L = \int_{t_1}^{t_2} \sqrt{(x')^2 + (y')^2}\, dt.$$

If smooth curve is given in the polar coordinates by the equations $\rho=\rho(\varphi)$, $\alpha\le\varphi\le\beta$, then the arc length is equal to $L=\int\limits_{\alpha}^{\beta}\sqrt{\rho^2+(\rho')^2}\,d\varphi$.

Example. $y=ln\,sin\,x$, $x\in\left[\frac{\pi}{3};\frac{\pi}{2}\right]$, $L-?$

Solution. $y'=\frac{cos\,x}{sin\,x}$; $L=\int\limits_{\frac{\pi}{3}}^{\pi/2}\sqrt{1+ctg^2x}dx=$

$$=\int\limits_{\pi/3}^{\pi/2}\frac{dx}{sin\,x}=ln\left|\frac{1}{sin\,x}-ctgx\right|\Bigg|_{\pi/3}^{\pi/2}=ln|1-0|-ln\left|\frac{2}{\sqrt{3}}-\frac{1}{\sqrt{3}}\right|=$$

$$=0-ln\frac{1}{\sqrt{3}}=+ln\sqrt{3}=\frac{1}{2}ln\,3 \text{ (length units).}$$

Example. To calculate the length of one arc of cycloid $x=t-sin\,t$; $y=1-cos\,t$.

Solution.

$$L=\int\limits_0^{2\pi}\sqrt{(1-cos\,t)^2+(sin\,t)^2}dt=\int\limits_0^{2\pi}\sqrt{1-2cos\,t+cos^2t+sin^2t}dt$$

$$=\int\limits_0^{2\pi}\sqrt{2-2cos\,t}dt=\sqrt{2}\int\limits_0^{2\pi}\sqrt{2sin^2\frac{t}{2}}dt=2\int\limits_0^{2\pi}sin\frac{t}{2}dt=$$

$$=-2\cdot 2cos\frac{t}{2}\Bigg|_0^{2\pi}=-4(cos\,\pi-cos\,0)=-4(-1-1)=8(length\ units)$$

Calculation of Solid Volume

1. The Calculation of Solid Volume according to the known areas of the cross-sections.

If the area of the solid section by the plane perpendicular to the X-axis can be represented as function of x, $S=S(x)$, $a\le x\le b$, then the solid volume is closed between the perpendicular X-axis planes $x=a$ and $x=b$ is calculated according to the formula

$$V=\int\limits_a^b S(x)dx.$$

2. <u>Calculation of the rotation solid volume</u>

If curvilinear trapezoid bounded by the curve $y=f(x)$ and the straight lines $x=a$, $x=b$, $y=0$, it is rotated round the X-axis then the rotation solid volume can be obtained by the formula

$$V_x=\pi\int_a^b y^2dx.$$

If a figure is bounded by the curves $y=f_1(x)$ and $y=f_2(x)$, $\left[0\le f_1(x)\le f_2(x)\right]$ and the straight lines $x=a$, $x=b$, it is rotated round the X-axis then the rotation solid volume is

$$V_x=\pi\int_a^b\left[f_2^2(x)-f_1^2(x)\right]dx.$$

Example. To calculate a solid volume obtained by the figure rotation round the X-axis bounded by the lines $y=x^2$, $x=y^2$ (Fig. 9).

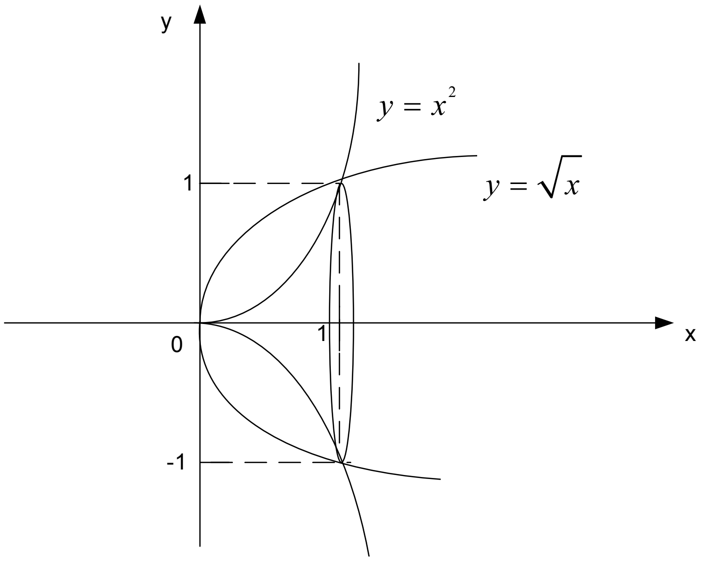


Fig.9

Solution. Let's find the points of intersection $x=y^2$; $y=x^2\Rightarrow y^2=x^4$.

Then $x=x^4$, $x\left(x^3-1\right)=0$; $x(x-1)\left(x^2+x+1\right)=0$; $x_1=0$; $x_2=1$

$$V_x\pi\int_0^1\left(x-x^4\right)dx=\pi\left[\frac{x^2}{2}-\frac{x^5}{5}\right]\Bigg|_0^1=\pi\left[\frac{1}{2}-\frac{1}{5}\right]=\frac{3\pi}{10}\quad\text{(cubic units).}$$

Calculation of rotation surface

If an arc of the smooth curve $y = f(x)$, $(a \le x \le b)$ is rotated round the X-axis then the rotation surface area is

$$S_x = 2\pi \int_a^b y\sqrt{1+(y')^2}\,dx$$

If a curve is given in the parametric form $x = x(t)$; $y = y(t)$, $(t_1 \le t \le t_2)$

Then $S_x = 2\pi \int_{t_1}^{t_2} y(t)\sqrt{(x')^2+(y')^2}\,dt$.

To calculate areas of the surfaces obtained by the rotation round the X-axis of the arc curves.

Example. $y = x^3$ from $x = 0$ to $x = 1$

Solution. $S_x = 2\pi \int_0^1 x^3\sqrt{1+(3x^2)^2}\,dx = 2\pi \int_0^1 (9x^4+1)^{\frac{1}{2}}\,d\frac{x^4}{4} =$

$$= \left| x^3dx = d\frac{x^4}{4} \right| = \frac{2\pi}{4\cdot 9}\int_0^1 (9x^4+1)^{\frac{1}{2}}\,d(9x^4+1) = \frac{\pi\cdot 2}{18}\frac{(9x^4+1)^{\frac{3}{2}}}{3}\Bigg|_0^1 =$$

$$= \frac{\pi}{27}\left[(9+1)^{\frac{3}{2}}-1\right] = \frac{\pi}{27}\left[10\sqrt{10}-1\right] \text{ (square units).}$$

Example. $x = t - sin\,t$; $y = 1 - cos\,t$, $0 \le t \le 2\pi$.

Solution. $S_x = 2\pi \int_0^{2\pi} (1-cos\,t)\sqrt{(1-cos\,t)^2+sin^2 t}\,dx =$

$$= 2\pi \int_0^{2\pi} (1-cos\,t)\sqrt{1-2\,cos\,t+cos^2 t+sin^2 t}\,dt =$$

$$= 2\pi \int_0^{2\pi} (1-cos\,t)\sqrt{2(1-cos\,t)}\,dt = 2\pi \int_0^{2\pi} 2\,sin^2\frac{t}{2}\sqrt{2\cdot 2\,sin^2\frac{t}{2}}\,dt =$$

$$= 2\cdot 8\pi \int_0^{2\pi} \left(1-cos^2\frac{t}{2}\right) sin\frac{t}{2}\,d\frac{t}{2} = -16\pi \int_0^{2\pi} \left(1-cos^2\frac{t}{2}\right) d\,cos\frac{t}{2} =$$

$$= -16\pi\left[cos\frac{t}{2} - \frac{cos^3\frac{t}{2}}{3}\right]_0^{2\pi} = -16\pi\left[-1+\frac{2}{3}\right] = \frac{64}{3}\pi \text{ (square units).}$$