The R-functions method application to solving mass transfer problems

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The task of mass transfer of the body of revolution with uniform translational flow is considered. A numerical method for its solution, based on the joint use of the R-functions structural method of V. L. Rvachev, Ukraine NAS academician, for constructing the structure of the boundary problem solution and the Galerkin-Petrov projection method for approximating the indeterminate components of the structure, is proposed.

Introduction

The tasks of mass transfer of bodies with uniform translational flow are the basis of many technological processes, associated with dissolution, extraction, evaporation, precipitation of colloids, etc. [1]. Such problems also find application in heat and power engineering, chemical and food technologies, geo- and astrophysical researches, environmental conservation.

In general case, the problem of the stationary mass transfer of the rotating body with a viscous incompressible fluid flow is reduced to solving the equation of the hydrodynamic flow over the surface and the equation for the concentration with the appropriate boundary conditions on the surface of the body and away from it. Accurately take into account the geometry of the region, and the boundary conditions (including the condition at infinity), it is possible, using the constructive apparatus of the V. L. Rvachev R-functions theory [2].

The purpose of this research is to develop a new method of solving the problem of mass transfer of the rotating body with uniform translational flow, based on the joint application of the R-functions structural method and the Galerkin-Petrov projection method.

This work is based on the acad. V. L. Rvachev R-functions method [2] and its applications to calculating the fluid flows in infinite simply connected domains with complex geometry [3].

Problem statement

Consider the mass transfer of the body of revolution with a viscous incompressible fluid. It is assumed that the Cartesian coordinate system (x, y, z) is introduced in the space and the streamlined body is formed by the rotation around the Oz axis of the figure, lying in the plane Oxz (the figure Ω is simply connected, finite and symmetrical in reference to the Oz axis). In addition, the fluid flow is supposed to be uniform, its speed is equal U_∞ beyond the body and it has the same direction that the Ox axis has. Such flows are conveniently treated in spherical coordinates (r, θ, φ), related to the Cartesian coordinate system (x, y, z) by the formulas:

$$x = r\cos\theta\cos\varphi,$$
$$y = r\cos\theta\sin\varphi,$$
$$z = r\sin\theta,$$
$$0 \le r < +\infty,\ 0 \le \theta \le \pi,\ 0 \le \varphi < 2\pi.$$

In axisymmetric problems in spherical coordinates r, θ, φ all values are independent of the coordinate φ and the third component of the fluid velocity is zero: $v_\varphi = 0$.

The components of the fluid velocity can be presented as [1, 4]

$$v_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta},\ v_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}, \tag{1}$$

where $\psi = \psi(r, \theta)$ is the stream function.

The mass transfer process is described by the equation for the concentration [1]

$$\Delta c = \mathrm{Pe}(\vec{v}\cdot\nabla)c, \tag{2}$$

where $c = c(r,\theta)$ is the concentration, $\Delta c = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial c}{\partial r}\right) + \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial c}{\partial\theta}\right)$, $(\vec{v}\cdot\nabla)c = v_r\dfrac{\partial c}{\partial r} + \dfrac{v_\theta}{r}\dfrac{\partial c}{\partial\theta}$, Pe — the Peclet number.

Considering (1), the equation (2) takes the form

$$\Delta c = \frac{\mathrm{Pe}}{r^2\sin\theta}\left(\frac{\partial\psi}{\partial\theta}\frac{\partial c}{\partial r} - \frac{\partial\psi}{\partial r}\frac{\partial c}{\partial\theta}\right).$$

Consider the following problem

$$\Delta c = \frac{\mathrm{Pe}}{r^2\sin\theta}\left(\frac{\partial\psi}{\partial\theta}\frac{\partial c}{\partial r} - \frac{\partial\psi}{\partial r}\frac{\partial c}{\partial\theta}\right) \text{ outside } \bar{\Omega}, \tag{3}$$

$$c|_{\partial\Omega} = 1, \tag{4}$$

$$c \to 0 \text{ as } r \to \infty. \tag{5}$$

The stream function $\psi = \psi(r,\theta)$ can be found, for example, as a solution of the linearized problem of the slow flow of a viscous incompressible fluid (the Stokes approximation) [1, 4]

$$E^2(E^2\psi) = 0 \text{ outside } \bar{\Omega}, \tag{6}$$

$$\psi|_{\partial\Omega} = 0, \left.\frac{\partial\psi}{\partial\mathbf{n}}\right|_{\partial\Omega} = 0, \tag{7}$$

$$\psi \sim \frac{1}{2}U_\infty r^2\sin^2\theta \text{ as } r \to \infty, \tag{8}$$

where $E^2 \equiv \dfrac{\partial^2}{\partial r^2} + \dfrac{\sin\theta}{r^2}\dfrac{\partial}{\partial\theta}\left(\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}\right)$, $\mathbf{n}$ is external to the $\partial\Omega$ normal.

Thus, the solution of the problem (3) — (5) consists of two stages:

1) the determination of the stream function as the solution of the task (6) — (8);

2) the solution of the task (3) — (5).

The solution method

For solving the problems, we will use the R-functions method of V. L. Rvachev, Ukraine NAS academician [2]: by constructive means of the R-functions theory we will build the structures for solving the boundary value problems, i.e. the bundles of functions, that exactly satisfy the boundary conditions.

Let outside $\bar{\Omega}$ a sufficiently smooth function, possessing next properties, is known :

$$1)\ \omega(r,\theta) > 0 \text{ outside } \bar{\Omega};\ 2)\ \omega(r,\theta) = 0 \text{ on } \partial\Omega;\ 3)\ \frac{\partial\omega(r,\theta)}{\partial\mathbf{n}} = -1 \text{ on } \partial\Omega.$$

We introduce a sufficiently smooth function $y = f_M(x)$, which satisfies following requirements:

$$\text{a) } f_M(0) = 0;\ \text{b) } f'_M(0) = 1;\ \text{c) } f'_M(0) \geq 0\ \ \forall x \geq 0;$$
$$\text{d) } f_M(x) \equiv 1\ \forall x \geq M\ (M = \mathit{const} > 0).$$

The conditions a) — d) are satisfied, for example, with function [3]

$$f_M(x) = \begin{cases} 1 - \exp\dfrac{Mx}{x-M}, & 0 \leq x < M; \\ 1, & x \geq M. \end{cases}$$

Obviously, that $f_M(x) \in C^\infty[0, +\infty)$. Denote

$$\omega_M(r, \theta) = f_M[\omega(r, \theta)]. \tag{9}$$

It is easy to verify that the function $\omega_M(r, \theta)$ satisfies to the conditions 1) — 3). Moreover, $\omega_M(r, \theta) \equiv 1$, if $\omega(r, \theta) \geq M$. This condition means that if the function $\omega(r, \theta)$ increases monotonically with distance from $\partial\Omega$, the function $\omega_M(r, \theta)$ of the form (9) is different from unity only in annular region $\{0 \leq \omega(r, \theta) < M\}$, that is contained in the exterior of $\bar{\Omega}$ and adjacent to the $\partial\Omega$.

In the paper [5] it is shown that for any choice of sufficiently smooth functions Φ_1 and Φ_2 ($\Phi_1 \cdot r^{-2} \to 0$ as $r \to +\infty$) the function of the form

$$\psi = \omega_M^2(\psi_0 + \Phi_1) + \omega_M^2(1 - \omega_M)\Phi_2 \tag{10}$$

exactly satisfies the boundary conditions (7) and the condition at infinity (8). Here $\psi_0 = \dfrac{1}{4}U_\infty(r - R)^2 \left(2 + \dfrac{R}{r}\right) \sin^2\theta$ — the Stokes solution for the problem of flow past a sphere of radius R (we consider that the sphere of radius R lies entirely inside of the streamlined body).

For approximating the indefinite components Φ_1 and Φ_2 it is proposed to use the Galerkin-Petrov method [6]. Functions Φ_1 and Φ_2 will present in a kind

$$\Phi_1 \approx \Phi_1^{m_1} = \sum_{k=1}^{m_1} a_k \varphi_k, \ \ \Phi_2 \approx \Phi_2^{m_2} = \sum_{j=1}^{m_2} b_j \tau_j,$$

where

$\varphi_k(r, \theta) = \left\{r^{1-k} J_k(\cos\theta),\ k = 2, 3, \ldots;\ r^{3-k} J_k(\cos\theta),\ k = 4, 5, \ldots\right\}$, — the complete system of partial solutions of the equation (6) regarding to the field $\{\omega(r, \theta) > 0\}$;

$\tau_j(r, \theta) = \left\{ r J_2(\cos\theta),\ J_3(\cos\theta),\ J_j(\cos\theta)\ \begin{matrix} r^j \\ r^{j+2}, \end{matrix} \right\}$, $j = 2, 3, \ldots,$ — the complete system of partial solutions of the equation (6) regarding to the field $\{\omega(r, \theta) < M\}$,

$J_n(\zeta)$ — the Gegenbauer functions [4].

We will determine the complete in relation to all plane sequence of functions

$$\begin{gathered}\{\omega_M^2(r, \theta)\ r^{1-k} J_k(\cos\theta),\ k = 2, 3, \ldots;\ \omega_M^2(r, \theta)\ r^{3-k} J_k(\cos\theta),\ k = 4, 5, \ldots; \\ \omega_M^2(r, \theta)(1 - \omega_M(r, \theta))\ r J_2(\cos\theta),\ \omega_M^2(r, \theta)(1 - \omega_M(r, \theta)) J_3(\cos\theta), \\ \omega_M^2(r, \theta)(1 - \omega_M(r, \theta))\ J_j(\cos\theta)\ \begin{matrix} r^j \\ r^{j+2}, \end{matrix}\ j = 2, 3, \ldots\}\end{gathered} \tag{11}$$

According to the Galerkin-Petrov method, the coefficients a_k ($k = 1, 2, \ldots, m_1$) and b_j ($j = 1, 2, \ldots, m_2$) we find from the condition of orthogonality of the discrepancy to the first $m_1 + m_2$ elements of the sequence (11). Thus, we will obtain an approximate solution of the problem (6) — (8).

Substituting the stream function in equation (3), we will solve the problem (3) — (5) also by the R-functions method.

Theorem. For any choice of sufficiently smooth functions Ψ_1 and Ψ_2 ($\Psi_1 \to 0$ as $r \to +\infty$) the function of the form

$$c = 1 - \omega_M + \omega_M \Psi_1 + \omega_M(1 - \omega_M)\Psi_2$$

exactly satisfies the boundary conditions (4) and (5).

For approximating the indefinite components Ψ_1 and Ψ_2 it is also proposed to use the Galerkin-Petrov method [6]. Functions Ψ_1 and Ψ_2 will present in a kind

$$\Psi_1 \approx \Psi_1^{m_3} = \sum_{k=1}^{m_3} \alpha_k \phi_k, \ \ \Psi_2 \approx \Psi_2^{m_4} = \sum_{j=1}^{m_4} \beta_j \gamma_j,$$

where

$\phi_k(r,\theta) = \left\{ r^{-k} \begin{matrix} \cos k\theta \\ \sin k\theta \end{matrix} \right\}$, $k = 1, 2, \ldots,$ — the complete system of partial solutions of the Laplace equation regarding to the field $\{\omega(r,\theta) > 0\}$;

$\gamma_j(r,\theta) = \left\{ r^{j} \begin{matrix} \cos j\theta \\ \sin j\theta \end{matrix} \right\}$, $j = 1, 2, \ldots,$ — the complete system of partial solutions of the Laplace equation regarding to the field $\{\omega(r,\theta) < M\}$.

We will determine the complete in relation to all plane sequence of functions

$$\left\{ \omega_M(r,\theta)\ r^{-k} \begin{matrix} \cos k\theta \\ \sin k\theta \end{matrix},\ \omega_M(r,\theta)\ (1-\omega_M(r,\theta))\ r^{j} \begin{matrix} \cos j\theta \\ \sin j\theta \end{matrix} \right\},\ k, j = 1, 2, \ldots \qquad (12)$$

According to the Galerkin-Petrov method, the coefficients α_k ($k = 1, 2, \ldots, m_3$) and β_j ($j = 1, 2, \ldots, m_4$) we find from the condition of orthogonality of the discrepancy to the first $m_3 + m_4$ elements of the sequence (12). Thus, we will obtain an approximate solution of the problem (3) — (5).

Conclusion

The method for calculating the mass transfer of the rotating body with uniform translational flow, based on the joint use of the R-functions structural method and the Galerkin-Petrov projection method, which differs from the known methods universality (the algorithm does not change with changes in the geometry of the area) and the fact that the structure of the solution accurately account for all the boundary conditions of the problem, has been proposed for the first time in the paper. The developed method allows to conduct the mathematical modeling of various technological, physical and mechanical processes.

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